International Journal of Technical Innovation in Modern Engineering & Science (IJTIMES)
Impact Factor: 5.22 (SJIF-2017), e-ISSN: 2455-2585
Volume 5, Issue 05, May-2019

STUDY OF SPECTRAL THEORY AND EIGEN FUNCTIONS EXPENSIONS FOR A PAIR OF SECOND ORDER MATRIX DIFFERENTIAL EQUATION

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ABSTRACT

The present paper concerned with spectral theory and eigen functions expansions for a pair of second order matrix differential equation. The theory of eigenfunction expansions associated with the second order differential equations dates as far back as the time of Sturm and Liouville. The discussion on the systems of simultaneous differential equations of first orderhowever, appears to have been initiated by Bocher in 1902.

Key Words: Matrix differential operator, convergence theorem, bounded variation.

1. INTRODUCTION

The theory of eigenfunctions expansions associated with the second order differential equations dates as far back as the time of Sturm and Liouville. The discussion on the systems of simultaneous differential equations of first order, however, appears to have been initiated by Bocher in 1902, and subsequently has been considered by Schlesinger, Eurwitz, Gamp, Birkoff and Longer, Bliss, Titchmarsh, Conte and Sangren, Boos and Sangren and other authors. Hilbert took up the discussions on a pair of simultaneous differential equations of the second order and Whybura, Kamke,.Lidskii, Levin, Kodaira. Coddinton and Levinson, Chakravarty, Bhagat, Tiwari studied problems with two (or more) simultaneous second order differential equations and advanced the theory to a large extent. In what follows we sketch in brief only a few previous works on zeros of eigenfunctions and related works. Spectra of differential equations and Inverse problems explained separately. It appears that the theory on the zeros of solutions of differential equations in a given interval was dealt with first by Sturm, a complete account of which and its subsequent development may be found in a Parismonograph by Bocher (1917). We however outline some of the developments of the theory in brief.

We consider the differential system

$$(M+\lambda)\phi = 0; \qquad 0 \leq x < \infty \#(1.1.1)$$

where M stands for the matrix differential operator given by

$$M \equiv \begin{pmatrix} \frac{d^2}{dx^2} - p(x) r(x) \\ r(x) \frac{d^2}{dx^2} - q(x) \end{pmatrix}$$

ϕ is a vector represented by a column matrix

$$\phi = \begin{pmatrix} u \\ v \end{pmatrix}$$

and λ is a parameter real of complex.
We assume the following conditions to be satisfied:

(i) p(x), q(x), r(x) are all real - valued and continuous functions in $0 \leq x \leq \infty$

(ii) p(x), q(x), r(x) are all $L[0,\infty)$. We suppose that any solution $\phi(x) = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}$ of the system satisfies the two linearly independent boundary conditions at x=0, viz :

$$S_{jl}u(0)+a_{j2}u^{'}(0)+a_{j3}v(0)a_{j4}v^{'}(0)=0,(j=1,2)\#(1.1.2)$$

where

(a) $a_{jk}\{j=1,2;k=1,2,3,4\}$ are real-valued constants;

(b) the set $\{a_{1k}\}$ is linearly independent of the set $\{a_{2k}\}$;

(c) $a_{14}a_{23}-a_{24}a_{13}+a_{12}a_{21}-a_{11}a_{22}=0$ (1.1.3)

Following Bhagat [3], the bilinear concomitant $[\phi \quad \theta]$ of two vectors $\phi=\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$ and $\theta=\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}$ is defined by

$$\phi=\phi_1^{'}\theta_1-\phi_1\theta_2^{'}+\phi_1^{'}\theta_2-\phi_2\theta_2^{'}$$

If ϕ and θ are any two solution of the system (1.1.1) for the same value of λ, then $[\phi \quad \theta]$ is a function of λ, then $[\phi \quad \theta]$ is a function of λ, real for real λ (See Bhagat (3)).
Let

$$\phi_j(x_2\lambda)=\phi_j(0|x_2\lambda)=\begin{pmatrix}u_j(0|x_2\lambda)\\ v_j(0|x_2\lambda)\end{pmatrix} \qquad (j=1,2)$$

by the boundary-condition vector then (1.1.2) and (1.1.2) can be written as

$$\left[\phi(x_2\lambda)\phi_j(0|x,\lambda)\right]=0 \quad (j=1,2)\#(1.1.4)$$

and

$$[\phi_1\phi_2]=0\#(1.1.5)$$

The vectors

$$\theta_k(x_2\lambda)=\theta_k(0|x_2\lambda)=\begin{pmatrix}x_k(0|x_2;\lambda)\\ y_k(0|x_2;\lambda)\end{pmatrix} \qquad (k=1,2)$$

which take real constant values (independent of λ) at x=0 are defined by the relations

$$\phi_j\theta_k=\delta_{jk}=[\theta_1\theta_2]=0\ (1\ \le j,k\le 2)\#(1.1.6)$$

(See Bhagat [5]).
It has been shown by Bhagat in [4] that the system (1.1.1.) has at least a pair of solutions belonging to $L^2[0.\infty]$ which are given by

$$\psi_r(x,\lambda)=\theta_r(x,\lambda)+\sum_{s=1}^{2}m_{rs}((\lambda)\,\phi_s(x,\lambda),(r=1,2)\#(1.1.7)$$

The $m_{rs}(\lambda)$ $(1\le r,\ s\le 2)$ are analytic functions of λ regular in either of the half plane im$\lambda>0$ and im$\lambda<0_2$ and $\overline{m_{rs}(\lambda)}=m_{rs}(\bar{\lambda})$. It is also proved that

$$\left[\phi_j(0|x_2\lambda)\psi_r(x,\lambda)\right]=\delta_{jr^2}(1\le j,r\le 2)(1.1.8)$$

(See Bhagat [4]).

2.THE GREEN'S MATRIX

$$G(x,y;\lambda)=\begin{pmatrix}G_{11}G_{21}\\ G_{12}\quad G_{22}\end{pmatrix}$$

for the system (1.1.1) is given by

$$G(x,y;\lambda)=\begin{pmatrix}\psi_{11}(x,\lambda)\psi_{21}(x,\lambda)\\ \psi_{12}(x,\lambda)\psi_{22}(x,\lambda)\end{pmatrix}\cdot\begin{pmatrix}\mathrm{u}_1(\mathrm{y},\lambda)\ \mathrm{v}_1(\mathrm{y},\lambda)\\ \mathrm{u}_2(\mathrm{y},\lambda)\mathrm{v}_2(\mathrm{y},\lambda)\end{pmatrix}y\in[0,x]$$

$$=\begin{pmatrix}\mathrm{u}_1(\mathrm{y},\lambda) & \mathrm{u}_2(\mathrm{y},\lambda)\\ \mathrm{v}_1(\mathrm{y},\lambda) & \mathrm{v}_2(\mathrm{y},\lambda)\end{pmatrix}\cdot\begin{pmatrix}\psi_{11}(y,\lambda)\psi_{12}(y,\lambda)\\ \psi_{21}(y,\lambda)\psi_{21}(y,\lambda)\end{pmatrix};y\in(x,\infty)\#(1.2.1)$$

We shall use the notations and results of Bhagat [3]

3. INTEGRAL EQUATIONS

$\phi_j(x,\lambda)$ $(j=1,2)$ satisfy the system of integral equations

$$\left.\begin{aligned} u_j(x,\lambda) &= u_j(0)\cos\mu x + \frac{1}{\mu}u_j'(0)\sin\mu x + \frac{1}{\mu}\int_0^x \{p(y)u_j(y,\lambda) - r(y)\,v_j(y,\lambda)\}\sin\mu(x-y)\,dy \\ v_j(x,\lambda) &= v_j(0)\cos\mu x + \frac{1}{\mu}v_j'(0)\sin\mu x + \frac{1}{\mu}\int_0^x \{q(y)v_j(y,\lambda) - r(y)\,u_j(y,\lambda)\}\sin\mu(x-y)\,dy \end{aligned}\right\}\#(j=1,2)$$

(1.3.1)

where $\lambda = \mu^2$.

We have from [5, ξ3] for large x, if $\mu = \sigma + it$, $t \geq 0$ and $|\mu| \geq \rho$

$$u_j(y,\lambda), v_j(x,\lambda) = 0\,(e^{tx}), (j = 1,2)\#(1.3.2)$$

$$\left.\begin{aligned} u_j(x,\lambda) &= e^{-1\mu x}\{M_{j1}(\lambda) + 0(1)\} \\ v_j(x,\lambda) &= e^{-1\mu x}\{M_{j2}(\lambda) + 0(1)\} \end{aligned}\right\}, \quad (j=1,2)\#(1.3.3)$$

where

$$\left.\begin{aligned} M_{j1}(\lambda) &= \frac{1}{2}u_j(0) - \frac{1}{2i\mu}u_j'(0) - \frac{1}{2i\mu}\int_0^\infty e^{\frac{1}{\mu y}}\{p\,(y)u_j(y,\lambda) - r(y)v_j(y,\lambda)\}dy, \\ M_{j2}(\lambda) &= \frac{1}{2}v_j(0) - \frac{1}{2i\mu}v_j'(0) - \frac{1}{2i\mu}\int_0^\infty e^{\frac{1}{\mu y}}\{q\,(y)v_j(y,\lambda) - r(y)u_j(y,\lambda)\}dy \end{aligned}\right\}, \quad (j=1,2)\#(1.3.4)$$

Also, from [6; ξ2) for $|\mu| \geq |\mu_o|$

$$\left.\begin{aligned} u_j(x,\lambda) &= u_j(0)\cos\mu x + 0\left\{\frac{e^{|tx|}}{|\mu|}\right\} \\ v_j(x,\lambda) &= v_j(0)\cos\mu x + 0\left\{\frac{e^{|tx|}}{|\mu|}\right\} \end{aligned}\right\}, \quad (j=1,2)\#(1.3.5)$$

4. SPECIAL SOLUTIONS

In this section we obtain two independent solutions of (1.1.1) which are small when imaginary part of λ is large and positive.

Consider the system of integral equations

$$\left.\begin{aligned} X_j(x) &= e^{|\mu x|} + \frac{1}{2i\mu}\int_0^x e^{1\mu(x-y)}\{p\,(y)X_j(y) - r(y)Y_j(y)\}dy + \\ &+ \frac{1}{2i\mu}\int_x^\infty e^{1\mu(y-x)}\{p\,(y)\,X_j\,(y) - r(y)Y_j(y)\}dy, \\ Y_j(x) &= e^{|\mu x|} + \frac{1}{2i\mu}\int_0^x e^{1\mu(x-y)}\{q\,(y)Y_j(y) - r(y)X_j(y)\}dy + \\ &+ \frac{1}{2i\mu}\int_x^\infty e^{1\mu(y-x)}\{p\,(y)\,X_j\,(y) - r(y)Y_j(y)\}dy, \end{aligned}\right\}, \quad (j=1,2)\#(1.4.1)$$

where $\lambda = \mu^2$

Differentiating (1.4.1) twice it can be verified (formally) that $\beta_j(x) = \begin{pmatrix} X_j(x) \\ Y_j(x) \end{pmatrix}$, (j = 1, 2) satisfy (1.1.1).

The solutions of (1.4.1) can be obtained by the method of successive approximation as follows:

Let

$$X_{j1}(x) = e^{i\mu x}, Y_{j1}(x)e^{i\mu x}, (j = 1,2)\#(1.4.2)$$

and for $n \geq 1$

$$\left.\begin{aligned} X_{jn+1}(x) &= e^{1\mu x}\frac{1}{2i\mu}\int_{\infty}^{x} e^{1\mu(x-y)}\{\mathrm{p}\ (y)\mathrm{X_{jn}}(y) - r\ (y)\mathrm{Y}_{jn}(y)\}\, dy + \\ &+\frac{1}{2i\mu}\int_{x}^{\infty} e^{1\mu(y-x)}\{\mathrm{p}\ (y)\mathrm{X_{jn}}(y) - r\ (y)\mathrm{Y}_{jn}(y)\}dy, \\ Y_{jn+1}(x) &= e^{1\mu x}\frac{1}{2i\mu}\int_{\infty}^{x} e^{1\mu(x-y)}\{\mathrm{q}\ (y)\mathrm{X_{jn}}(y) - r\ (y)\mathrm{X}_{jn}(y)\}\, dy + \\ &+\frac{1}{2i\mu}\int_{x}^{\infty} e^{1\mu(y-x)}\{\mathrm{q}\ (y)\mathrm{Y_{jn}}(y) - r\ (y)\mathrm{X}_{jn}(y)\}dy, \end{aligned}\right\}, \qquad (j = 1,2)\#(1.4.3)$$

Since $p(x), q(x), r(x)$ are all $L\ [0,\infty]$, so we suppose that

$$\mathrm{J} = \mathrm{Max}\left\{\int_0^\infty |\mathrm{p(x)}|\mathrm{dx},\quad \int_0^\infty |\mathrm{q(x)}|\mathrm{dx},\quad \int_0^\infty |\mathrm{r(x)}|\mathrm{dx}\right\}\#(1.4.4)$$

Then

$$\mathrm{X_{j2}(x) - X_{jl}(x)} = \frac{e^{1\mu x}}{2i\mu}\left[\int_0^x \{\mathrm{p\ (y) - r\ (y)}\}\mathrm{dy} + \int_x^\infty \{\mathrm{p\ (y) - r\ (y)}\}\mathrm{e}^{2\mathrm{i}\mu\,(\mathrm{y-x})}\mathrm{dy}\right]$$

or,

$$\left|\left|\mathrm{X_{j2}(x) - X_{jl}(x)}\right| \le \frac{e^{-\mathrm{tx}}}{|\mu|}\mathrm{J},\right|, \qquad (j = 1,2)\#(1.4.5)$$

Similarly

$$\left|\left|\mathrm{Y_{j2}(x) - Y_{jl}(x)}\right| \le \frac{e^{-\mathrm{tx}}}{|\mu|}\mathrm{J},\right|, \qquad (j = 1,2)\#(1.4.6)$$

Hence by using (2.4.5) and (2.4.6) we have

$$\left|\mathrm{X_{j3}(x) - X_{j2}(x)}\right| \le \frac{e^{-tx}}{|\mu|^2}\mathrm{J}^2, \qquad (j = 1,2)\#(1.4.7)$$

and

$$\left|\mathrm{Y_{j3}(x) - X_{j2}(x)}\right| \le \frac{e^{-tx}}{|\mu|^2}\mathrm{J}^2, (j = 1,2)\#(1.4.8)$$

and so on.

Therefore, it follows that if $|\mu| > \mathrm{J}$, the series

$$\sum_{\mathrm{n}=1}^{\infty}\left(\mathrm{X_{jn+1}(x) - X_{jn}(x)}\right)$$

and

$$\sum_{\mathrm{n}=1}^{\infty}\left(\mathrm{X_{jn+1}(x) - X_{jn}(x)}\right)$$

are convergent.

Let $\mathrm{X_j((x) = lim_{n\to\infty} X_{jn}(x)}$ and $\mathrm{Y_j(x) = lim_{n\to\infty} Y_{jn}(x)}$

Now for every n

$$\begin{aligned} \left|\mathrm{X_{jn}(x)}\right| &\le \left|\mathrm{X_{j1}(x)}\right| + \left|\mathrm{X_{j2}(x) - X_{j1}(x)}\right| + \ \ldots. + \left|\mathrm{X_{jn}(x) - xX_{jn-1}(x)}\right| \\ &\le e^{-tx}\{1 + (\mathrm{J}/|\mu|) + \cdots + (\mathrm{J}/|\mu|^{\mathrm{N}-1}\} \\ &= e^{-tx}\cdot\frac{[1-(\mathrm{J}/|\mu|)^{\mathrm{N}}]}{[1-(\mathrm{J}/|\mu|)]} \le \frac{e^{-tx}}{\left(1-\frac{\mathrm{J}}{|\mu|}\right)} \end{aligned}$$

so, for $n \to \infty$

$$\left|X_j(x)\right| = \lim_{n\to\infty} |X_{jn}(x)| \le \frac{e^{-tx}}{\left(1-\frac{J}{|\mu|}\right)}, \qquad (j=1,2)\#(1.4.9)$$

Similarly

$$\left|Y_j(x)\right| = \lim_{n\to\infty} |Y_{jn}(x)| \le \frac{e^{-tx}}{\left(1-\frac{J}{|\mu|}\right)}, \qquad (j=1,2)\#(1.4.10)$$

Therefore, by dominated convergence, it follows that the limit operation can betaken under the integral sign and that $\beta_j(x)(j=1,2)$ satisfy the equations (1.4.1) and hence (1.1.1).
Now for a fixed μ or μ in the bounded part of region $|\mu| > J$, (2.4.1) gives

$$X_j(x) = e^{i\mu x} + \frac{e^{i\mu x}}{2i\mu}\int_0^\infty \{p(y)X_j(y) - r(y)Y_j(y)\}e^{-1\mu y}\,dy - \frac{e^{i\mu x}}{2i\mu}\int_x^\infty \{p(y)X_j(y) - r(y)Y_j(y)\}e^{-1\mu y}\,dy + + \frac{e^{i\mu x}}{2i\mu}\int_x^\infty \{p(y)X_j(y) - r(y)Y_j(y)\}e^{-1\mu y}(y-2x)dy$$

The first integral is convergent and the last two integrals tends to zero as $x \to \infty$, therefore

$$X_j(x) = e^{i\mu x}\{C_{j1}(\lambda) + \circ(1)\}, \qquad (j=1,2)\#(1.4.11)$$

Where

$$C_{j1}(\lambda) = 1 + \frac{1}{21\mu}\int_0^\infty e^{-1\mu y}\{p(y)K_j(y) - r(y)Y_j(y)\}\,dy, \qquad (j=1,2)\#(1.4.12)$$

Similarly

$$T_j(x) = e^{i\mu y}\{C_{j2}(\lambda) + 0(1)\}, \qquad (j=1,2)\#(1.4.13)$$

Where

$$C_{j2}(\lambda) = 1 + \frac{1}{2i\mu}\int_0^\infty e^{-i\mu y}\{q(y)Y_j(y) - r(y)X_j(y)\}\,dy, \qquad (j=1,2)\#(1.4.14)$$

5.THE TWO LINEARLY INDEPENDENT SOLUTIONS

The two linearly independent solutions of the system (1.1.1) which are $L^2[0,\infty]$, are given by (1.1.7). Now by (2.3.3), $u_j(x,\lambda), v_j(x,\lambda)$ (j=1,2) are large when the imaginary part of λ is large and positive and

$$\left.\begin{aligned} M_{j1}(\lambda) &\sim -\frac{u_j(0)}{2}, \text{ if } u_j(0) \ne 0 \\ &\sim -\frac{u_j(0)}{2i\mu}, \text{ if } u_j(0) = 0 \\ M_{j2}(\lambda) &\sim -\frac{v_j(0)}{2}, \text{ if } v_j(0) \ne 0 \\ &\sim -\frac{v_j(0)}{2i\mu}, \text{ if } v_j(0) = 0. \end{aligned}\right\}, \qquad (j=1,2)\#(1.5.1)$$

So $\phi_j(x,\lambda), j=(1,2)$ are not $L^2[0,\infty)$. But from (1.4.11) and (1.4.13), we see that $X_j(x,\lambda), Y_j(x,\lambda)$, $(j = 1,2)$are small when imaginary part of λ is large and positive. Therefore, we conclude that$\psi_r(x,\lambda)$are linearly independent.
Then

$$\psi_r(x,\lambda) = \sum_{s=1}^{2} K_{rs}(\lambda)\beta_s(x,\lambda) + \sum_{s=1}^{2} \ell_{rs}(\lambda)\phi_s(x,\lambda), (r=1,2)\,\#(1.5.2)$$

Since $\beta_r(x,\lambda)$ $(r=1,2)$ are $L^2[0,\infty)$, but $\phi_j(x,\lambda)$ $(j=1,2)$ are not $L^2[0,\infty)$, therefore $\ell_{rs}(\lambda) = 0$ $(1 \le r,s \le 2)$.
Hence

$$\psi_r(x,\lambda) = \sum_{s=1}^{2} K_{rs}(\lambda)\beta_s(x,\lambda), \qquad (r=1,2)\,\#(1.5.3)$$

From the asymptotic formulae (2.3.3), (2.4.11) and (2.4.13) we obtain, as $x \to \infty$

$$\left.\begin{array}{l} u_j'(x,\lambda) \sim -i\mu e^{-i\mu x} M_{j1}(\lambda) \\ v_j'(x,\lambda) \sim -i\mu e^{-i\mu x} M_{j1}(\lambda) \\ X_j'(x,\lambda) \sim -i\mu e^{-i\mu x} c_{j1}(\lambda) \\ Y_j'(x,\lambda) \sim -i\mu e^{-i\mu x} c_{j2}(\lambda). \end{array}\right\}, \qquad (j = 1,2) \#(1.5.4)$$

where dashes denote differentiation with respect to x. Using (1.3.3), (1.4.11), (1.4.13), (1.5.3) and (1.5.4), we obtain from (1.1.8).

$$\left.\begin{array}{r} K_{11}(M_{11}C_{11} + M_{12}C_{12}) + K_{12}(M_{11}C_{21} + M_{12}C_{11}) + \dfrac{1}{2i\mu} = 0, \\ K_{11}(M_{21}C_{11} + M_{22}C_{12}) + K_{12}(M_{21}C_{21} + M_{22}C_{22}) = 0. \\ K_{21}(M_{11}C_{11} + M_{12}C_{12}) + K_{12}(M_{11}C_{21} + M_{12}C_{11}) = 0. \\ K_{21}(M_{21}C_{11} + M_{22}C_{12}) + K_{22}(M_{21}C_{21} + M_{22}C_{22}) + \dfrac{1}{2i\mu} = 0. \end{array}\right\} \#(1.5.5)$$

From (2.5.5) we get

$$\left.\begin{array}{l} K_{11} = \dfrac{(M_{21}C_{21} + M_{22}C_{22})}{2i\mu(M_{11}M_{22} - M_{12}M_{21})\,(C_{12}C_{21} - C_{11}C_{22})} \\ K_{12} = \dfrac{(M_{21}C_{11} + M_{22}C_{12})}{2i\mu(M_{11}M_{22} - M_{12}M_{21})\,(C_{12}C_{21} - C_{11}C_{22})} \\ K_{21} = \dfrac{(M_{11}C_{21} + M_{12}C_{22})}{2i\mu(M_{11}M_{22} - M_{12}M_{21})\,(C_{12}C_{21} - C_{11}C_{22})} \\ K_{22} = \dfrac{(M_{11}C_{11} + M_{12}C_{12})}{2i\mu(M_{11}M_{22} - M_{12}M_{21})\,(C_{12}C_{21} - C_{11}C_{22})} \end{array}\right\} \#(1.5.6)$$

6. CONVERGENCE THEOREM

If $f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$ be a real-valued vector of bounded variation in $0 \le x < \infty$ and be $L^2[0,\infty)$ and $L^2[0,\infty)$ and $\lambda \ne$ and eigenvalue of the system (2.1.1), then

$$\frac{1}{2}\{f(x+0) + f(x-0)\} = -\frac{1}{\pi i} \lim_{R \to \infty} \int_{-R+i\epsilon}^{R+i\epsilon} \Phi(x,\lambda), d\lambda \,\#(1.6.1)$$

uniformly for $0 < \epsilon \le 1$, where

$$\Phi(x,\lambda) = \begin{pmatrix} \Phi_1(x,\lambda) \\ \Phi_2(x,\lambda) \end{pmatrix} = \int_0^\infty G(x,y;\ \lambda) f(y) dy \,\#(1.6.2)$$

We prove convergence theorem for $\Phi_1(x,\lambda)$ because similar result holds for $\Phi_2(x,\lambda)$. Now we write $\Phi_1(x,\lambda)$ as

$$\begin{aligned} \Phi_2(x,\lambda) = {} & \psi_{11}(x,\lambda) \int_0^x \phi_1^T f(y)\ dy + \psi_{21}(x,\lambda) \int_0^x \phi_1^T f(y)\ dy + \\ & + u_1(x,\lambda) \int_x^\infty \psi_1^T(y,\lambda)\, f(y) dy + u_2(x,\lambda) \int_x^\infty \psi_2^T(y,\lambda)\, f(y) dy + \\ & + \psi_{11}(x,\lambda) \int_0^x u_1(y,\lambda)\, f_1(y) dy + u_1(x,\lambda) \int_x^\infty \psi_{11} f(y,\lambda) f_1(y)\ dy + \\ & + \psi_{21}(x,\lambda) \int_0^x u_2(y,\lambda)\, f_1(y) dy + u_2(x,\lambda) \int_x^\infty \psi_{21} f(y,\lambda) f_1(y)\ dy + \\ & + \psi_{11}(x,\lambda) \int_0^x v_1(y,\lambda)\, f_2(y) dy + \psi_{21}(x,\lambda) \int_0^x v_2(y,\lambda)\, f_2(y) dy + \end{aligned}$$

$$+u_1(x,\lambda)\int_x^\infty \psi_{12}(y,\lambda)f_2(y)dy \;+\; u_2(x,\lambda)\int_x^\infty \psi_{22}(y,\lambda)f_2(y)dy\,.$$
$$= A+B+C+D+E+F\ (say) \#(1.6.3)$$

where

$$A=\ \psi_{11}(x,\lambda)\int_0^x u_1(y,\lambda)\,f_1(y)dy\ +u_1(x,\lambda)\int_x^\infty \psi_{11}(y,\lambda)\,f_1(y)dy$$

$$B=\ \psi_{21}(x,\lambda)\int_0^x u_2(y,\lambda)\,f_1(y)dy\ +u_2(x,\lambda)\int_x^\infty \psi_{21}(y,\lambda)\,f_1(y)dy$$

$$C=\ \psi_{11}(x,\lambda)\int_0^x v_1(y,\lambda)\,f_2(y)dy$$

$$D=\ \psi_{21}(x,\lambda)\int_0^x v_2(y,\lambda)\,f_2(y)dy$$

$$E=u_1(x,\lambda)\int_x^\infty \psi_{12}(y,\lambda)\,f_2(y)dy$$

$$F=u_2(x,\lambda)\int_x^\infty \psi_{22}(y,\lambda)\,f_2(y)dy$$

We evaluate A, the other terms can be evaluated in the same way.
Now

$$A=\ \psi_{11}(x,\lambda)\int_0^x u_1(y,\lambda)\,f_1(y)dy\ +u_1(x,\lambda)\int_x^\infty \psi_{11}(y,\lambda)\,f_1(y)dy$$

$$=\psi_{11}(x,\lambda)\left[\int_0^{x-\delta}+\int_{x-\delta}^{x}.\right]+u_1(x,\lambda)\left[\int_x^{x+\delta}+\int_{x+\delta}^{\infty}.\right]$$

$$=A_1+A_2+A_3+A_4\ ,(\text{say}).$$

For $|\mu|>J$, we have from (1.5.3) and (1.5.6)

$$|\psi_{11}(y,\lambda)|\le\frac{|M_{22}(\lambda)||e^{i\mu y}|}{2|\mu|\{|M_{11}(\lambda)M_{22}(\lambda)-M_{12}(\lambda)M_{21}(\lambda)|\}}<\frac{\propto\ e^{-ty}}{|\mu|}\#(1.6.4)$$

where $\propto$ is a constant.

Therefore, using (1.3.2) and (1.6.4), we have

$$A_4=\circ\{\frac{e^{tx}}{|\mu|}\int_{x+\delta}^{\infty}\{e^{-ty}|f_1(y)|dy\}=\circ\left\{\frac{e^{tx}}{|\mu|}\right\}\#(1.6.5)$$

The integral of (1.6.5) round the semicircle tends to zero as R→ ∞ for any fixed ?. Similar arguments hold for A_1 also. Now we consider A_3. For fixed x or in a finite interval, from (1.4.1)

$$\left|X_j(x,\lambda)-\ e^{i\mu x}\right|=\frac{1}{2i\mu}\left|\int_0^x e^{i\mu(x-y)}\{p(y)X_j(y,\lambda)-r(y)Y_j(y,\lambda)\}dy\ +\right.$$
$$\left.+\frac{1}{2i\mu}\int_x^\infty e^{i\mu(y-x)}\{p(y)X_j(y,\lambda)-r(y)Y_j(y,\lambda)\}dy\right|$$

$$\Rightarrow \left|X_j(x,\lambda) - e^{i\mu x}\right| \le \frac{e^{-tx}}{2|\mu|}(J + \ldots\ldots) < \frac{\beta e^{-tx}}{|\mu|}, (j = 1,2), \qquad say\#(1.6.6)$$

Similarly

$$\left|Y_j(x,\lambda) - e^{i\mu x}\right| < \frac{\beta e^{-tx}}{|\mu|}, (j = 1,2), \qquad say\#(1.6.7)$$

Also from (1.4.12) and (1.4.14), we have

$$C_{j1}(\lambda) = \left\{1 + 0\left(\frac{1}{|\mu|}\right)\right\}, \quad (j = 1, 2)\#(1.6.8)$$

$$C_{j2}(\lambda) = \left\{1 + 0\left(\frac{1}{|\mu|}\right)\right\}, \quad (j = 1, 2)\#(1.6.9)$$

Similarly, from (1.3.4), we have

$$\left.\begin{aligned} M_{j1}(\lambda) &= \left\{\frac{u_j(0)}{2} + 0\left(\frac{1}{|\mu|}\right)\right\} \\ M_{j2}(\lambda) &= \left\{\frac{v_j(0)}{2} + 0\left(\frac{1}{|\mu|}\right)\right\} \end{aligned}\right\}, (j = 1,2)\#(1.6.10)$$

Therefore, by using (1.6.8) and (1.6.9), (1.4.11) and (1.4.11) and (1.4.13) can be written respectively as

$$\left.\begin{aligned} X_j(x,\lambda) &= e^{i\mu x}\left\{1 + 0\left(\frac{1}{|\mu|}\right)\right\} \\ Y_j(x,\lambda) &= e^{i\mu x}\left\{1 + 0\left(\frac{1}{|\mu|}\right)\right\} \end{aligned}\right\}, (j = 1,2)\#(1.6.11)$$

Now using (1.6.10) and (1.6.11), (1.5.3) and (1.5.6) give

$$\psi_{11}(x,\lambda) = \frac{v_2(0)e^{i\mu x}\left\{1 + 0\left(\frac{1}{|\mu|}\right)\right\}}{i\mu[v_1(0)u_2(0) - u_1(0)v_2(0)]}\#(1.6.12)$$

Thus, from the first result of (1.3.5) and (1.6.12), we get

$$A_3 = \frac{u_1(0)v_2(0)\cos\mu x}{i\mu[v_1(0)u_2(0) - u_1(0)v_2(0)]}\int_x^{x+\delta} e^{i\mu x} f_1(y)dy + 0\left\{\frac{e^{|tx|}}{\frac{1}{|\mu|}^2}\int_x^{x+\delta} e^{-ty} f_1(y)dy\right\}$$

The last term of A_3 is

$$0\left\{\frac{1}{|\lambda|}\int_x^{x+\delta} f_1(y)dy\right\}$$

and the integral of this round the semicircle is

$$0\left\{\int_x^{x+\delta} f_1(y)dy\right\}$$

which can be made as small as we please by properly choosing δ. The first term in A_3 can be written as

$$\frac{v_2(0)u_1(0)(e^{i\mu x} + e^{i\mu x})}{2i\mu[v_1(0)u_2(0) - u_1(0)v_2(0)]}\int_x^{x+\delta} e^{i\mu x} f_1(y)dy$$

The term involving $e^{i\mu x}$ also gives a zero limit by similar arguments. The remaining term is the same as in the case of an ordinary Fourier series. Similar arguments also hold for A_2. Hence, we conclude that in the bounded variation case

$$\lim_{R\to\infty}\int_{-R+i\epsilon}^{R+i\epsilon} A d\lambda = \frac{1}{2}\pi i\frac{u_1(0)v_2(0)\{f_1(x+o) + f_2(x+o)\}}{[v_2(0)u_1(0) - u_1(0)v_2(0)]}$$

Similarly

$$\lim_{R\to\infty}\int_{-R+i\in}^{R+i\in} Bd\lambda = \frac{1}{2}\pi i\frac{v_1(0)u_2(0)\{f_1(x+o)+f_1(x+o)\}}{[v_1(0)u_2(0)-u_1(0)v_2(0)]}$$

$$\lim_{R\to\infty}\int_{-R+i\in}^{R+i\in} Cd\lambda = \frac{1}{2}\pi i\frac{v_2(0)v_2(0)\{f_1(x+o)+f_2(x+o)\}}{[v_2(0)u_1(0)-u_1(0)v_2(0)]}$$

$$\lim_{R\to\infty}\int_{-R+i\in}^{R+i\in} Dd\lambda = \frac{1}{2}\pi i\frac{v_1(0)v_2(0)f_2(x-0)}{[v_1(0)u_2(0)-u_1(0)v_2(0)]}$$

$$\lim_{R\to\infty}\int_{-R+i\in}^{R+i\in} Ed\lambda = \frac{1}{2}\pi i\frac{u_1(0)u_2(0)f_2(x+0)}{[v_1(0)u_2(0)-u_1(0)v_2(0)]}$$

$$\lim_{R\to\infty}\int_{-R+i\in}^{R+i\in} Fd\lambda = \frac{1}{2}\pi i\frac{v_1(0)u_2(0)f_2(x+0)}{[v_1(0)u_2(0)-u_1(0)v_2(0)]}$$

Thus, we have

$$\lim_{R\to\infty}\int_{-R+i\in}^{R+i\in} \Phi_1(x,\lambda)\, d\lambda = -\frac{1}{2}\pi i\,\{f_1(x+0)+f_1(x-0)\}$$

If f (x) is continuous, then

$$f_1(x) = -\frac{1}{\pi\, i}\lim_{R\to\infty}\int_{-R+i\in}^{R+i\in} \Phi_1(x,\lambda)\, d\lambda \quad \#(1.6.13)$$

Similarly

$$f_2(x) = -\frac{1}{\pi\, i}\lim_{R\to\infty}\int_{-R+i\in}^{R+i\in} \Phi_2(x,\lambda)\, d\lambda \quad \#(1.6.14)$$

The above regulation is true uniformly for0 $< \epsilon \le 1$

7.THE MATRIX

$$K(\lambda) = \left(k_{rs}(\lambda)\right) = (\lim_{\nu\to 0}\int_0^{\lambda} - \mathrm{im}\, m_{rs}(\mu+i\nu)d\mu\,)\ \#(1.7.1)$$

exists for all real λ is a function of bounded variation and

$$K(\lambda) = \frac{1}{2}[K(\lambda+0) + K(\lambda-0)]\#(1.7.2)$$

Also

$$\lim_{\nu\to 0}\int_0^{\lambda} -\mathrm{im}\psi_r(x,\mu+i\nu)d\mu = \sum_{S=1}^{2}\int_0^{\lambda}\phi_s(x,\mu)dk_{rs}(\mu)\ \#(1.7.3)$$

8. EXPANSIONS

In this section we investigate the behavior of the integrals (1.6.13) and (1.6.13) and (1.6.14) as $\epsilon \to 0$. Here we discuses (1.6.13) and the same arguments will apply to (1.6.14). First of all, we show that (1.6.13) can be replaced by

$$f_1(x) = -\lim_{R\to\infty}\left[\frac{1}{\pi}\int_{-R-i\epsilon}^{R+i\epsilon} \mathrm{im}\ \Phi_1(x,\lambda)d\lambda\right] \#(1.8.1)$$

Since $\Phi_1(x,\lambda)$ is analytic in the upper and lower half planes, it follows from the convergence theorem that

$$f_1(x) = -\frac{1}{\pi i}\lim_{R\to\infty}\int_{-R-i\epsilon}^{R+i\epsilon} \Phi_1(x,\lambda)d\lambda \#(1.8.2)$$

Let $\lambda = s - i\epsilon = \bar{\lambda}$-2 $i\epsilon$, ϵ being fixed. Then from (2.8.2), we have

$$f_1(x) = -\frac{1}{\pi i}\lim_{R\to\infty}\int_{-R+i\epsilon}^{R+i\epsilon} \Phi_1(x,\bar{\lambda}-2i\,\epsilon)d\bar{\lambda} = \frac{1}{\pi i}\lim_{R\to\infty}\int_{-R+i\epsilon}^{R+i\epsilon} \Phi_1(x,\bar{\lambda})d\bar{\lambda} \#(1.8.3)$$

Adding (1.6.13) and (1.8.3) we have

$$2f_1(x) = -\frac{1}{\pi i}\lim_{R\to\infty}\int_{-R+i\epsilon}^{R+i\epsilon} \Phi_1(x,\lambda) - \Phi_1(x,\bar{\lambda})\ d\bar{\lambda} = \frac{2}{\pi i}\lim_{R\to\infty}\int_{-R+i\epsilon}^{R+i\epsilon} [\mathrm{im}\ \Phi_1(x,\lambda)]d\lambda$$

because $\mathrm{im}\ \Phi(x,\bar{\lambda}) = \Phi(x,\lambda)$. The proves (1.8.1)

We know that $\phi_j(x,\lambda), \theta_j(x,\lambda)(j=1,2)$ are analytic functions of λ and are real for real λ, it follows that each of

$$\mathrm{im}(u_j), \mathrm{im}(v_j), \mathrm{im}(x_j), \mathrm{im}(y_j) = 0\ (\epsilon)\#(1.8.4)$$

as $\epsilon \to 0$, Therefore, for x, y in the fixed interval

$$\mathrm{im}\{\psi_{r1}(x,\lambda)u_j(x,\lambda)\psi_{r1}(y,\lambda)\} = 0\ (\epsilon), (1 \le r, j \le 2)\#(1.8.5)$$

and

$$\mathrm{im}\{\psi_{11}(x,\lambda)v_1(y,\lambda) - u_1(x,\lambda) - \psi_{12}(y,\lambda) + \psi_{21}(x,\lambda)v_2(y,\lambda) - u_2(x,\lambda) - \psi_{22}(y,\lambda)\} = \circ\ (\epsilon)\#(1.8.6)$$

Now

$$\mathrm{im}\left[-\frac{1}{\pi}\int_{-R+i\epsilon}^{R+i\epsilon} \Phi_1(x,\lambda)\ d\lambda\right]$$

may be put in the form

$$\begin{aligned}\mathrm{im}\Bigg[-\frac{1}{\pi}\int_{-R+i\epsilon}^{R+i\epsilon}\Bigg\{&\int_0^\infty \big(u_1(x,\lambda)\psi_{11}(y,\lambda) + u_2(x,\lambda)\psi_{21}(y,\lambda)\big)f_1(y)dy + \\ &+\int_0^\infty \big(u_1(x,\lambda)\psi_{12}(y,\lambda) + u_2(x,\lambda)\psi_{22}(y,\lambda)\big)f_2(y)\,dy + \\ &+\int_0^x \big(u_1(y,\lambda)\psi_{11}(x,\lambda) - u_1(x,\lambda)\psi_{11}(y,\lambda)\big)f_1(y)\,dy + \\ &+\int_0^x \big(u_2(y,\lambda)\psi_{21}(x,\lambda) - u_2(x,\lambda)\psi_{21}(y,\lambda)\big)f_1(y)\,dy + \\ &+\int_0^x \big(\psi_{11}(x,\lambda)v_1(y,\lambda) - u_1(x,\lambda)\psi_{12}(y,\lambda) + \psi_{21}(x,\lambda)v_2(y,\lambda) \\ &- u_2(x,\lambda)\psi_{22}(y,\lambda)\big)f_2(y)dy\Bigg\}d\lambda\end{aligned}$$

(1.8.7)

Using (2.8.4), (2.8.6) and (2.8.6), (2.8.7) becomes, for fixed R and $\epsilon \to 0$

$$\mathrm{im}\left[-\frac{1}{\pi}\int_{-R+i\epsilon}^{R+i\epsilon}\left\{\int_0^\infty\left(u_1(x,\lambda)\psi_{11}(y,\lambda)+u_2(x,\lambda)\psi_{21}(y,\lambda)\right)f_1(y)dy + \int_0^\infty\left(u_1(x,\lambda)\psi_{12}(y,\lambda)+u_2(x,\lambda)\psi_{22}(y,\lambda)\right)f_2(y)\,dy\right\}d\lambda\right]+\circ(\epsilon) \tag{1.8.8}$$

Now

$$\int_{-R+i\epsilon}^{R+i\epsilon}\mathrm{im}\ u_1(x,\lambda)\,d\lambda\int_0^\infty \mathrm{re}\,\psi_{22}(y,\lambda)f_1(y)dy=\circ\ (\epsilon)\left\{\int_{-R}^{R}da\int_{-R}^{R}|\psi_{11}(y,s-i\epsilon)f_1(y)|dy\right\}$$

$$=0\ (\epsilon)\left\{\int_{-R}^{R}\left(\int_0^\infty|\psi_{11}(y,s-i\epsilon)f_1(y)|dy\right)da\right\}^{\frac{1}{2}}=\circ\left(\epsilon^{\frac{1}{2}}\right),$$

using schwarz's inequality and (2.7.4)

Similarly

$$\int_{-R}^{R}\mathrm{re}\{u_1(x,s-i\in)u_1(x,s)\}ds\int_0^\infty \mathrm{im}\,\psi_{11}(y,\lambda)f_1(y)dy=0\ \left(\in^{\frac{1}{2}}\right)$$

Therefore (2.8.8) becomes

$$\frac{1}{\pi}\left[\int_{-R}^{R}u_1(x,s)ds\int_0^\infty \mathrm{im}\psi_{11}(y,s-i\in)f_1(y)dy+\int_{-R}^{R}u_2(x,s)ds\int_0^\infty \mathrm{im}\psi_{21}(y,s-i\in)f_1(y)\,dy+\right.$$

$$\left.+\int_{-R}^{R}u_2(x,s)ds\int_0^\infty \mathrm{im}\psi_{12}(y,s-i\in)f_1(y)\,dy+\int_{-R}^{R}u_2(x,s)ds\int_0^\infty \mathrm{im}\psi_{22}(y,s-i\in)f_2(y)\,dy\right]+\circ\left(\epsilon^{\frac{1}{2}}\right)$$

$$=-\frac{1}{\pi}\left[\int_{-R}^{R}u_1(x,s)ds\int_0^\infty \mathrm{im}\psi_1^T(y,s-i\in)\,f(y)\,dy+\int_{-R}^{R}u_2(x,s)ds\int_0^\infty \mathrm{im}\psi_2^T(y,s-i\in)\,f(y)\,dy\right]+\circ\left(\epsilon^{\frac{1}{2}}\right) \tag{1.8.9}$$

Now let

$$\chi_r(y,\lambda)=\sum_{l=1}^{2}\int_0^\lambda \phi_\ell(y,s)dk_{r\ell}(s),\quad (r=1,2)\ \#(1.8.10)$$

and

$$g_r(\lambda)=\int_0^\infty \chi_r^T(y,\lambda)\ f(y)\,dy,\qquad (r=1,2)\#(1.8.11)$$

The integrals in (1.8.11) exist because f (x) is $L^2[0,\infty]$ and $\chi_r(y,\lambda)$ is $L^2[0,\infty]$ by (1.7.3)

$$\int_0^\lambda da\int_0^\infty \mathrm{im}\psi_r^T(y,s-i\epsilon)f(y)\,dy=\int_0^\infty f^T(y)f(y)\,dy\int_0^\lambda \mathrm{im}\psi_r(y,s-i\epsilon)\,da$$

$$\to -g_r(\lambda),\text{as }\epsilon\to 0$$

uniformly over a finite real λ range. Hence integrating the first term of (1.8.9) by parts, we have

$$-\frac{1}{\pi}\left[u_1(x,s)\int_0^s ds'\int_0^\infty \mathrm{im}\psi_1^T(y,s'-i\in)\,f(y)\,dy\right]_{-R}^{R}+\frac{1}{\pi}\left[\int_{-R}^{R}\frac{\partial u_1(x,s)}{\partial S}ds'\int_0^\infty \mathrm{im}\psi_1^T(y,s'-i\in)\,f(y)\,dy\right]$$

$$\to\frac{1}{\pi}[u_1(x,s)g_1(s)]_{-R}^{R}-\frac{1}{\pi}\int_{-R}^{R}\frac{\partial u_1(x,s)}{\partial S}g_1(s)\ ds,\#(1.8.12)$$

as $\in\to 0$. If $g_1(s)$ (r=1,2) are of bounded variation, is equal to the Stieltjes (1.8.12) integral

$$\frac{1}{\pi}\int_{-R}^{R} \mathrm{u}_1(\mathrm{x},\mathrm{s})\mathrm{dg}_1(\mathrm{s})$$

Therefore, (2.8.9) becomes

$$\frac{1}{\pi}\int_{-R}^{R} \mathrm{u}_1(\mathrm{x},\mathrm{s})\mathrm{dg}_1(\mathrm{s}) + \int_{-R}^{R} \mathrm{u}_2(\mathrm{x},\mathrm{s})\mathrm{dg}_1(\mathrm{s}) \#(1.8.13)$$

In those cases which involve continuous spectrum, it has been shown by Bhagat that

$$\lim_{t\to 0} \mathrm{im}\,[\mathrm{m_{jj}}] = \frac{\sum_{\ell=1}^{2}\{\mathrm{c}_{\mathrm{k}\ell}^2 + \mathrm{c}_{\mathrm{k}\ell}^2\}}{\lambda^{\frac{1}{2}}\left(\mathrm{A}^2(\lambda) + \mathrm{B}^2(\lambda)\right)} \#(1.8.14)$$

(when j = 1, k = 2 and when j=2, k=1)

$$\lim_{t\to 0} \mathrm{im}\,[\mathrm{m}_{12}] = \lim_{t\to 0} \mathrm{im}\,[\mathrm{m}_{21}] = -\frac{c_{11}c_{21} + c_{12}c_{22} + d_{11}d_{21} + d_{12}d_{22}}{\lambda^{\frac{1}{2}}\left(\mathrm{A}^2(\lambda) + \mathrm{B}^2(\lambda)\right)} \#(1.8.15)$$

where

$$\left.\begin{aligned}
A(\lambda) &= c_{21}c_{12} - d_{21}d_{12} - c_{11}c_{22} + d_{11}d_{22}\\
\mathrm{B}\,(\lambda) &= c_{21}d_{12} + c_{12}d_{21} - c_{11}d_{22} - c_{21}d_{11},\\
\mathrm{c_{j1}}(\lambda) &= \mathrm{u_j}(0) - \frac{1}{\mu}\int_0^\infty \{\mathrm{p(y)u_j(y)} - \mathrm{r\,(y)v_j(y)}\}\sin\mu\,\mathrm{y}\ \ \mathrm{dy},\\
\mathrm{d_{j1}}(\lambda) &= \frac{\mathrm{u_j}(0)}{\mu} + \frac{1}{\mu}\int_0^\infty \{\mathrm{p(y)u_j(y)} - \mathrm{r\,(y)v_j(y)}\}\cos\mu\,\mathrm{y}\ \ \mathrm{dy},\\
\mathrm{c_{j2}}(\lambda) &= \mathrm{v_j}(0) - \frac{1}{\mu}\int_0^\infty \{\mathrm{q(y)v_j(y)} - \mathrm{r\,(y)u_j(y)}\}\sin\mu\,\mathrm{y}\ \ \mathrm{dy},\\
\mathrm{d_{j2}}(\lambda) &= \frac{\mathrm{v}_j'(0)}{\mu} + \frac{1}{\mu}\int_0^\infty \{\mathrm{q(y)v_j(y)} - \mathrm{r\,(y)u_j(y)}\}\cos\mu\,\mathrm{y}\ \ \mathrm{dy},
\end{aligned}\right\} \#(1.8.16)$$

Now substituting the values of $\mathrm{dk_r(s)}(1 \le \mathrm{r},\ \ell \le 2)$in (1.8.10), we have

$$\chi_\mathrm{r}(\mathrm{y},\lambda) = \begin{pmatrix}\chi_{\mathrm{r}_1}(\mathrm{y},\lambda)\\ \chi_{\mathrm{r}_2}(\mathrm{y},\lambda)\end{pmatrix}$$

$$\int_0^\lambda \phi_r(y,s)\frac{\sum_{\ell=1}^{2}\{c_{k\ell}^2(s) + d_{k\ell}^2(s)\}}{s^{\frac{1}{2}}(A^2(s) + B^2(s))}\ ds -$$

$$-\int_0^\lambda \phi_k(y,s)\,\frac{\mathrm{c}_{11}(\mathrm{s})\mathrm{c}_{21}(\mathrm{s}) + \mathrm{c}_{12}(\mathrm{s})\mathrm{c}_{22}(\mathrm{s}) + \mathrm{d}_{11}(\mathrm{s})\mathrm{d}_{21}(\mathrm{s}) + \mathrm{d}_{12}(\mathrm{s})\mathrm{d}_{22}(\mathrm{s})}{s^{\frac{1}{2}}\left(A^2(s) + B^2(s)\right)}ds \#(1.8.17)$$

Using (1.8.11) and (1.8.17), (1.8.13) becomes

$$\frac{1}{\pi}\sum_{r=1}^{2}\chi_\mathrm{r}(\mathrm{y},\lambda)\int_0^\infty \phi_\mathrm{r}^\mathrm{T}(\mathrm{y},\lambda)\ \mathrm{f(y)}\ \ \mathrm{dy}$$

where the integration over λ is over the interval of continuous spectrum.

In the interval $-\mathrm{R} < \lambda < 0\ \ \mathrm{g_r}(\lambda)(r = 1,2)$ are constant except for a finite number of discontinuities at the poles of $\mathrm{m_{rs}}(\lambda)\quad (1 \le \mathrm{r,s} \le 2)$. Hence the associated expansion is a series.

9. CONCLUSIOS

Hence we find the integration is the interval of continuous spectrum and the associated expansion is a series.

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