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Stability and Bifurcation in a Fractional Order Brusselator Model And Its Discretization

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Abstract* - The Periodic temporal oscillation and formation of a spatial pattern in chemical reactions are major part of nonlinear chemical dynamics. Although the history of oscillatory reactions in chemical kinetics is long enough, it was less than quarter century ago that chemical oscillator was characterized. The key features of such oscillating reactions are the auto catalysis. In this present work, a fractional order Brusselator model is proposed. First we prove the existence and uniqueness of the solutions of fractional order dynamical system and with discretization process its discrete version is obtained. Local stability of the fixed point of Brusselator model has been studied in both commensurate and incommensurate fractional order system. Also we discuss the bifurcation parameters; it is proved that the discrete fractional order system undergoes a flip (period doubling) and Neimark – Sacker bifurcation at the interior (coexistence) fixed point. Finally numerical examples are provided to validate the analytical results and the rich dynamical nature of the discretized system is exhibited.

Keywords – Fractional order system, Discretization, Fixed point, Brusselator model, Stability, Bifurcations and Chaos.

1. INTRODUCTION

The Oscillatory chemical reaction modeled by Brusselator illustrates the trimolecular steps involved in these reactions. These chemical reactions are very important role to play their similarities with biological and neuronal networks [4]. The oscillatory system was first observed by Boris Belousov while studying Kreb's cycle. The periodic color changes due to bromate, citric acid and cerium catalyst mixture in a solution of sulfuric acid. The dynamics of oscillating chemical reactions emerged as new area of study after six decades of Boris Belousov work. The commonly known Brusselator model is an auto catalytic reaction in which the products obtained from a reaction becomes reactants of another reaction regenerating the original reactants. Equilibrium and homogeneity state for most reactions are attained quickly whereas a Belousov-Zhabotinsky reaction remains in a non-equilibrium state for a long time with pattern formations and oscillations. The chemical reaction of Brusselator model is similar to Belousov-Zhabotinsky (or B-Z) reaction [8, 9]. The oscillatory chemical behaviour of Brusselator reaction that was commonly studied by Prigogine and Lefever in 1968 [15]. In general, the mechanism for the Brusselator chemical reaction which is given by the following steps:

$$A \rightarrow X,\; 2X + Y \rightarrow 3X,\; B + X \rightarrow Y + D,\; X \rightarrow E.$$

Under these conditions, the two species x and y are proportional to the autocatalytic species X and Y. The differential equations given in dimensionless form for these two species are:

$$\frac{dx}{dt} = a - (1+b)x + x^2 y$$
$$\frac{dy}{dt} = bx - x^2 y$$

For this analysis, the parameters a and b are positive constants and the chemical reactants A and B are in vast excess so that their concentrations do not change with time.

Nowadays, the fractional order differential equations are gaining momentum in the research community due to their ability to describe various non-linear phenomena. The fractional order systems are more suitable realistic and the physical systems can be modeled more accurately with the assistance of fractional calculus [1, 10, 11]. In the last few years, with the development of numerical techniques to solve the fractional order differential equations and integrals, the applications of fractional calculus in different branches of science and engineering is growing rapidly [12, 13]. In this work we have considered the following two dimensional nonlinear fractional order differential equations:

$$\begin{aligned} {}_{t_0}^{C}D_t^{\varepsilon_1} x(t) &= a - (1+b)x + x^2 y \\ {}_{t_0}^{C}D_t^{\varepsilon_2} y(t) &= bx - x^2 y \end{aligned} \tag{1}$$

with the initial conditions $x(t_0) = x_{t_0} > 0$ and $y(t_0) = y_{t_0} > 0$, where ${}_{t_0}^{C}D_t^{\varepsilon_1}$ and ${}_{t_0}^{C}D_t^{\varepsilon_2}$ are in the sense of the Caputo fractional derivative which satisfies $0 < \varepsilon_{1,2} \le 1$ and $t_0 \ge 0$ is the initial time. All the parameters of system (1) are positive constants. The Caputo fractional derivative of order ε is given by ${}_{t_0}^{C}D_t^{\varepsilon} f(t) = \frac{1}{\Gamma(n-\varepsilon)} \int_{t_0}^{t} (t-s)^{n-\varepsilon-1} f^{(n)}(s)ds$, where $\Gamma(.)$ is the Gamma function, $t \ge t_0$ and n is a positive integer such that $n-1 < \varepsilon \le n$.

2. PRELIMINARIES

In this section, we give some basic definitions and theorems to introduce local stability of the fractional-order differential equation (1).

Definition 1: We consider the nonlinear fractional orders autonomous system

$$
\begin{aligned}
{}_{t_0}^{C}D_t^{\varepsilon_1} y_1(t) &= f_1(y_1, y_2, ..., y_n), \\
{}_{t_0}^{C}D_t^{\varepsilon_2} y_2(t) &= f_2(y_1, y_2, ..., y_n), \\
&\vdots \\
{}_{t_0}^{C}D_t^{\varepsilon_n} y_n(t) &= f_n(y_1, y_2, ..., y_n),
\end{aligned} \tag{2}
$$

with the initial values $y_1(0) = y_{01}, y_2(0) = y_{02}, \cdots, y_n(0) = y_{0n}$, where $0 < \varepsilon_i \le 1$, for $i = 1, 2, \cdots, n$. If $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_n = \varepsilon$, system (2) is called a commensurate order ε. Otherwise it is called an incommensurate order system.

Definition 2: The constant $\left(y_1^{eq}, y_2^{eq}, \cdots, y_n^{eq}\right)$ is a fixed point of the fractional order system (2), iff $f_i\left(y_1^{eq}, y_2^{eq}, \cdots, y_n^{eq}\right) = 0$, where $i = 1, 2, \cdots, n$.

Theorem 1: ([1, 6]) Let us consider the commensurate nonlinear fractional order autonomous system (2) with $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_n = \varepsilon \le 1$. The fixed points of system (2), y^{eq} are locally asymptotically stable (LAS) if all eigenvalues λ_i of the Jacobian matrix J evaluated at the fixed points

$$
J = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{bmatrix}, \ \alpha_{ij} = \frac{\partial f_i}{\partial y_j} \mid y^{eq}, \text{for } i, j = 1, 2, \cdots, n \text{ satisfy: } \left|\arg(\lambda_i)\right| > \frac{\varepsilon\pi}{2}.
$$

Theorem 2: ([5]) Consider an incommensurate nonlinear fractional order system (2) where ε_i's for $i = 1, 2, \cdots, n$ are rational numbers between 0 and 1. Let M be the least common multiple (LCM) of the denominators u_i of ε_i's, where $\varepsilon_i = \frac{v_i}{u_i}, \left(u_i, v_i\right) = 1$ (the greatest common divisor (GCD) of u_i and v_i is 1), $u_i, v_i \in \mathbf{Z}^+$, for $i = 1, 2, \cdots, n$ and $\gamma = \frac{1}{M}$.

Then, the fixed points of the fractional order system (2) are LAS if and only if all the roots λ's of the equation

$$
\det \begin{pmatrix} \lambda^{M\varepsilon_1} - \alpha_{11} & -\alpha_{12} & \cdots & -\alpha_{1n} \\ -\alpha_{21} & \lambda^{M\varepsilon_2} - \alpha_{22} & \cdots & -\alpha_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ -\alpha_{n1} & -\alpha_{n2} & \cdots & \lambda^{M\varepsilon_n} - \alpha_{nn} \end{pmatrix} = 0 \text{ satisfy: } \left|\arg(\lambda)\right| > \frac{\gamma\pi}{2}. \tag{3}
$$

3. EXISTENCE AND UNIQUENESS

This section, we establish the sufficient condition for existence and uniqueness of the solutions of the nonlinear fractional order system (1) with commensurate order $\varepsilon_1 = \varepsilon_2 = \varepsilon$ for $0 < \varepsilon \le 1$.

Theorem 3: The sufficient condition for the existence and uniqueness of the solutions of the commensurate nonlinear fractional order system (1) in the region $\Omega \times (0, L]$ with initial values $U(0) = U_0$ and $t \in (0, L]$ is

$$
\beta = \frac{L^{\varepsilon}}{\Gamma(\varepsilon+1)} \max\left\{1 + 2b + 4\xi^2; 2\xi^2\right\} < 1.
$$

Proof: The nonlinear fractional order system (1) can be written as

$$
{}_{t_0}^{C}D_t^{\varepsilon} U(t) = F(U(t)), \ t \in (0, L], \ U(0) = U_0,
$$

where

$$
U = \begin{bmatrix} x \\ y \end{bmatrix}, \ U_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \ F(U) = \begin{bmatrix} a - (1+b)x + x^2 y \\ bx - x^2 y \end{bmatrix}.
$$

Define the maximum norm as follows $\|N\| = \max_{t \in (0, L]} |N(t)|$.

The norm of the matrix $M = \left[m_{i,j}[t]\right]$ is defined by $\|M\| = \max_j \sum_{i=1} \left|m_{i,j}\right|$ [3].

Now we give the existence and uniqueness of the solution which are studied in the region $\Omega\times(0,L]$, where

$$\Omega=\left\{(x,y)\in\mathbb{R}_+^2:\max\left(|x|,|y|\le\xi\right)\right\}.$$

Therefore, the solution of fractional order system (1) is given as follows

$$T(U)=U=U_0+\frac{1}{\Gamma(\varepsilon)}\int_0^t (t-s)^{\varepsilon-1}F(U(s))ds.$$

So

$$T(U_1)-T(U_2)=\frac{1}{\Gamma(\varepsilon)}\int_0^t (t-s)^{\varepsilon-1}\left(F(U_1(s))-F(U_2(s))\right)ds.$$

We get the following inequality

$$\left|T(U_1)-T(U_2)\right|\le\frac{1}{\Gamma(\varepsilon)}\int_0^t \left|(t-s)^{\varepsilon-1}\left(F(U_1(s))-F(U_2(s))\right)\right|ds$$

Now

$$\left\|T(U_1)-T(U_2)\right\|\le\frac{L^\varepsilon}{\Gamma(\varepsilon+1)}\max\left\{1+2b+4\xi^2;2\xi^2\right\}\left\|U_1-U_2\right\|$$

$$\left\|T(U_1)-T(U_2)\right\|\le\beta\left\|U_1-U_2\right\|$$

where

$$\beta=\frac{L^\varepsilon}{\Gamma(\varepsilon+1)}\max\left\{1+2b+4\xi^2;2\xi^2\right\}.$$

The Lipschitz condition is satisfied by $T(U)$. If $\beta<1$, then the mapping $U=T(U)$ is a contraction mapping. This gives the existence and uniqueness of the commensurate nonlinear fractional order system (1).

4. Asymptotic Stability of Fixed Point of System (1)

In this section, we investigate the asymptotic stability of the fixed point of system (1) by using Theorems 1 and 2. In order to find the fixed point of system (1), we set ${}_{t_0}^{C}D_t^{\varepsilon_1}x(t)=0$ and ${}_{t_0}^{C}D_t^{\varepsilon_2}y(t)=0$. Then, the nonlinear fractional order system (1) has one fixed point as follows:

- The interior (coexisting) fixed point $E_1=\left(a,\frac{b}{a}\right)$.

The dynamical behavior of the interior fixed point can be studied by computing the eigenvalues of the Variation matrix V of fractional order system (1), namely, $V(x,y)=\begin{bmatrix}2xy-(1+b) & x^2\\ b-2xy & -x^2\end{bmatrix}$. (4)

4.1. The commensurate nonlinear fractional-order system

Consider the commensurate nonlinear fractional-order system (1) with $\varepsilon_1=\varepsilon_2=\varepsilon$ for $0<\varepsilon\le1$. The Variation matrix V of system (1) evaluated at the interior fixed point E_1 is $V\left(a,\frac{b}{a}\right)=\begin{bmatrix}b-1 & a^2\\ -b & -a^2\end{bmatrix}$ and the corresponding characteristic polynomial of the fixed point E_1 is given by $\lambda^2+\mu_1\lambda+\mu_2=0$, (5)
where $\mu_1=a^2-b+1$ and $\mu_2=a^2>0$. Then, it is easy to find by Theorem 1 that a sufficient condition for the local asymptotic stability at the interior fixed point E_1 is $\left|\arg(\lambda_j)\right|>\frac{\varepsilon\pi}{2}$, $j=1,2$. (6)

where λ_j is the solution of the characteristic equation (5) and is given by $\lambda_{1,2}=\frac{-\mu_1}{2}\pm\frac{\sqrt{\mu_1^2-4\mu_2}}{2}$.

Since $\mu_2>0$, $\lambda_{1,2}$ are negative real or complex conjugate with negative real part for $\mu_1>0$. In this case, the local stability condition (6) is equivalent to the Routh–Hurwitz conditions [2]. That is,

$$\mu_1^2\ge4\mu_2,\mu_1>0,\ \text{or}\ \mu_1^2<4\mu_2,\mu_1>0.$$

For $\mu_1<0$, the solutions λ_j are positive real or complex conjugate with positive real part. Thus, we have the following equivalent condition for (6)

$$\mu_1<0,4\mu_2>\mu_1^2,\left|\tan^{-1}\left(\frac{\sqrt{4\mu_2-\mu_1^2}}{\mu_1}\right)\right|>\frac{\varepsilon\pi}{2}.$$

Theorem 4: The interior fixed point E_1 of commensurate nonlinear fractional order system (1) is locally asymptotically stable if $\mu_1 < 0$, $4\mu_2 > \mu_1^2$ and $\varepsilon < \frac{2}{\pi}\tan^{-1}\frac{\sqrt{4\mu_2 - \mu_1^2}}{\mu_1}$.

Corollary 1: If $\mu_1 < 0$, $4\mu_2 > \mu_1^2$ and $\varepsilon > \frac{2}{\pi}\tan^{-1}\frac{\sqrt{4\mu_2 - \mu_1^2}}{\mu_1}$ then the interior fixed point E_1 of commensurate nonlinear fractional order system (1) is unstable.

4.2. An incommensurate nonlinear fractional-order system

Let us consider an incommensurate nonlinear fractional order system (1), $\varepsilon_1, \varepsilon_2 \in (0,1]$ and $\varepsilon_i = \frac{v_i}{u_i}$ with $(u_i, v_i) = 1$, $i = 1,2$. Then, the fixed point E_1 of system (1) is asymptotically stable if and only if all the roots λ's of the equation

$$\det\begin{pmatrix} \lambda^{M\varepsilon_1} - \alpha_{11} & -\alpha_{12} \\ -\alpha_{21} & \lambda^{M\varepsilon_2} - \alpha_{22} \end{pmatrix} = 0 \quad (7)$$

satisfy $|\arg(\lambda)| > \frac{\gamma\pi}{2}$, where M be the LCM of the denominators u_i, and $\gamma = \frac{1}{M}$.

5. Dynamical Behavior of Fractional Order System (1)

In this section, we analyze the fixed point and applying discretization process of a fractional - order system (1) with commensurate $\varepsilon_1 = \varepsilon_2 = \varepsilon$ for $0 < \varepsilon \le 1$ outlined in [10], we obtain the discrete fractional order system as follows:

$$\begin{aligned} x(t+1) &= x(t) + \frac{s^\varepsilon}{\Gamma(\varepsilon+1)}\left(a - (1+b)x(t) + x^2(t)y(t)\right) \\ y(t+1) &= y(t) + \frac{s^\varepsilon}{\Gamma(\varepsilon+1)}\left(bx(t) - x^2(t)y(t)\right) \end{aligned} \quad (8)$$

We next study the stability of fixed point of system (8). At the interior fixed point $E_1(x,y)$, the Variation matrix of the system (8) has the form:

$$V(x,y) = \begin{bmatrix} 1 + \frac{s^\varepsilon}{\Gamma(\varepsilon+1)}\left(2xy - (1+b)\right) & \frac{s^\varepsilon}{\Gamma(\varepsilon+1)}x^2 \\ \frac{s^\varepsilon}{\Gamma(\varepsilon+1)}(b - 2xy) & 1 - \frac{s^\varepsilon}{\Gamma(\varepsilon+1)}x^2 \end{bmatrix}$$

When E_1 exists, the Variation matrix at E_1 is given by $V(E_2) = \begin{bmatrix} 1 + \frac{s^\varepsilon}{\Gamma(\varepsilon+1)}(b-1) & \frac{s^\varepsilon}{\Gamma(\varepsilon+1)}a^2 \\ -\frac{s^\varepsilon}{\Gamma(\varepsilon+1)}b & 1 - \frac{s^\varepsilon}{\Gamma(\varepsilon+1)}a^2 \end{bmatrix}$.

Therefore, the eigenvalues of $V(E_2)$ are $\lambda_{1,2} = 1 + \frac{s^\varepsilon}{\Gamma(\varepsilon+1)}\left(\frac{(b-1-a^2) \pm \sqrt{\Delta}}{2}\right)$.

It is easy to see that $\lambda_{1,2}$ satisfy the equation $F(\lambda) = \lambda^2 + \mu_3\lambda + \mu_4 = 0$, (9)

where $\mu_3 = \frac{s^\varepsilon}{\Gamma(\varepsilon+1)}\left(a^2 - b + 1\right) - 2$, $\mu_4 = 1 - \frac{s^\varepsilon}{\Gamma(\varepsilon+1)}\left(a^2 - b + 1\right) + \left(\frac{s^\varepsilon}{\Gamma(\varepsilon+1)}\right)^2 a^2$ and $\Delta = a^2\left(a^2 - 2b - 2\right) + (b-1)^2$.

Using Jury's criterion [7], we have the necessary and sufficient condition for the local stability of the fixed point E_1 which are given in the following theorem.

Theorem 5: The interior fixed point $E_1 = \left(a, \frac{b}{a}\right)$ of the discrete fractional order system (8) satisfies the following conditions :

(i) it is a sink if one of the following conditions hold:

(i.a) $\Delta \ge 0$ and $\frac{2\left(\frac{s^\varepsilon}{\Gamma(\varepsilon+1)}(1-b) - 2\right)}{\frac{s^\varepsilon}{\Gamma(\varepsilon+1)}\left(\frac{s^\varepsilon}{\Gamma(\varepsilon+1)} - 2\right)} < a^2 < \frac{b-1}{1 - \frac{s^\varepsilon}{\Gamma(\varepsilon+1)}}$; **(i.b)** $\Delta < 0$ and $a^2 < \frac{b-1}{1 - \frac{s^\varepsilon}{\Gamma(\varepsilon+1)}}$;

(ii) it is a source if one of the following conditions hold:

(ii.a) $\Delta \geq 0$ and $a^2 > \max\left\{ \dfrac{2\left(\dfrac{s^\varepsilon}{\Gamma(\varepsilon+1)}(1-b)-2\right)}{\dfrac{s^\varepsilon}{\Gamma(\varepsilon+1)}\left(\dfrac{s^\varepsilon}{\Gamma(\varepsilon+1)}-2\right)}, \dfrac{b-1}{1-\dfrac{s^\varepsilon}{\Gamma(\varepsilon+1)}} \right\}$; **(ii.b)** $\Delta < 0$ and $a^2 > \dfrac{b-1}{1-\dfrac{s^\varepsilon}{\Gamma(\varepsilon+1)}}$;

(iii) it is non-hyperbolic if one of the following conditions hold:

(iii.a) $\Delta \geq 0$ and $a^2 = \dfrac{2\left(\dfrac{s^\varepsilon}{\Gamma(\varepsilon+1)}(1-b)-2\right)}{\dfrac{s^\varepsilon}{\Gamma(\varepsilon+1)}\left(\dfrac{s^\varepsilon}{\Gamma(\varepsilon+1)}-2\right)}$; **(iii.b)** $\Delta < 0$ and $a^2 = \dfrac{b-1}{1-\dfrac{s^\varepsilon}{\Gamma(\varepsilon+1)}}$;

(iv) it is a saddle for the other parameter values excluding those discussed in **(i) - (iii)**.

6. NUMERICAL SIMULATION

This section, numerical simulation are investigated to confirm our theoretical results of the systems (1) and (8) with the aid of time plots, phase portraits and bifurcation diagrams for various parameter values. Our numerical simulations will be divided into two parts: the first part will focus on the simulations of commensurate and incommensurate fractional order system (1), while the next part will concern the numerical simulations of the discrete fractional order system (8).

6.1 Simulations of Model (1)

Numerical result of the nonlinear fractional-order system (1) has the following form [14]:

$$x(t_m) = \left(a-(1+b)x(t_{m-1}) + x^2(t_{m-1})y(t_{m-1})\right)h^{\varepsilon_1} - \sum_{l=v}^{m} c_l^{(\varepsilon_1)} x(t_{m-l}),$$

$$y(t_m) = \left(bx(t_m) - x^2(t_m)y(t_{m-1})\right)h^{\varepsilon_2} - \sum_{l=v}^{m} c_l^{(\varepsilon_2)} y(t_{m-l}),$$

where T_{sim} is the simulation time, $m = 1,2,3,\cdots,N$ for $N = \left[T_{sim}/h\right]$, and $(x(0), y(0))$ is the start point (initial conditions).

6.1.1. Commensurate Fractional Order $(\varepsilon_1 = \varepsilon_2 = \varepsilon)$:

For a commensurate nonlinear fractional order system (1), the stability and instability of the interior fixed point E_1 is followed from Theorem 4 and Corollary 1. Let us consider the following set of parameter values $a = 0.66$, $b = 1.39$ with the fractional order $\varepsilon = 0.97$ and initial conditions $x_0 = 0.4$ and $y_0 = 0.6$. We obtain the interior (coexistence) fixed point $E_1 = (0.66, 2.1061)$ and the eigenvalues are $\lambda_{1,2} = -0.0228 \pm i0.6596$. Clearly, the stability condition $\varepsilon = 0.97 < \dfrac{2}{\pi}\left|\arg(\lambda_j)\right| = 0.9784$ is satisfied. Therefore, the fractional order system (1) has a stable fixed point at E_1, see Figure 1.

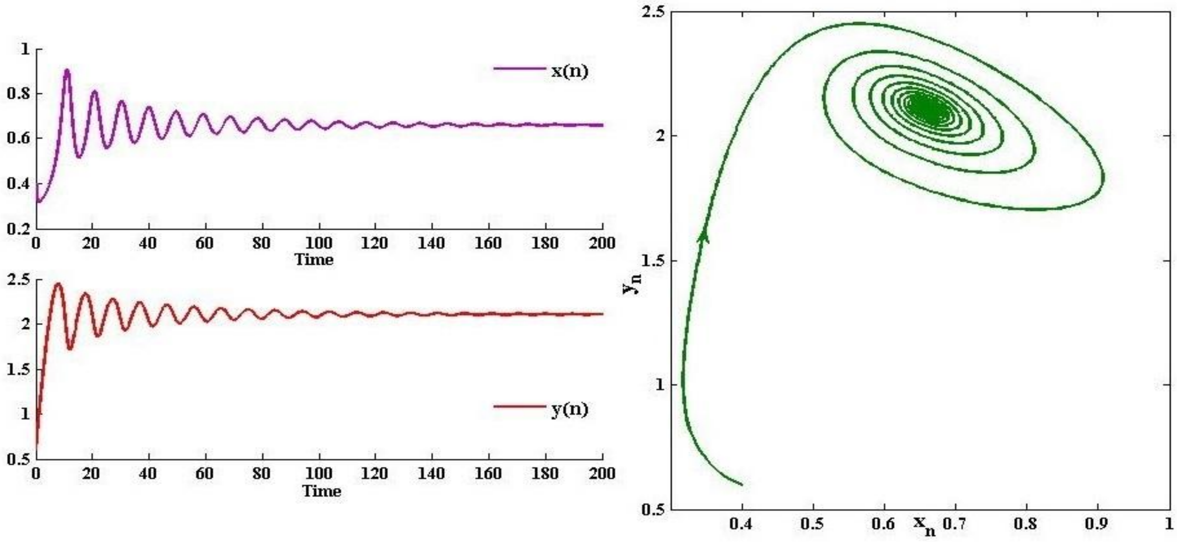

Figure 1: *Time series and Phase portrait are stable at E_1.*

Let us consider the parameter values $a = 0.956$, $b = 2.368$ with the fractional order $\varepsilon = 0.96$ and the initial conditions $x_0 = 0.4$ and $y_0 = 0.6$. We obtain the interior (coexistence) fixed point $E_1 = (0.9560, 2.4770)$ and the eigenvalues are $\lambda_{1,2} = 0.2270 \pm i0.9287$. Clearly, the instability condition $\varepsilon = 0.96 > \frac{2}{\pi}\left|\arg\left(\lambda_j\right)\right| = 0.8477$ is satisfied Therefore, the fractional order system (1) has an unstable fixed point at E_1, see Figure 2.

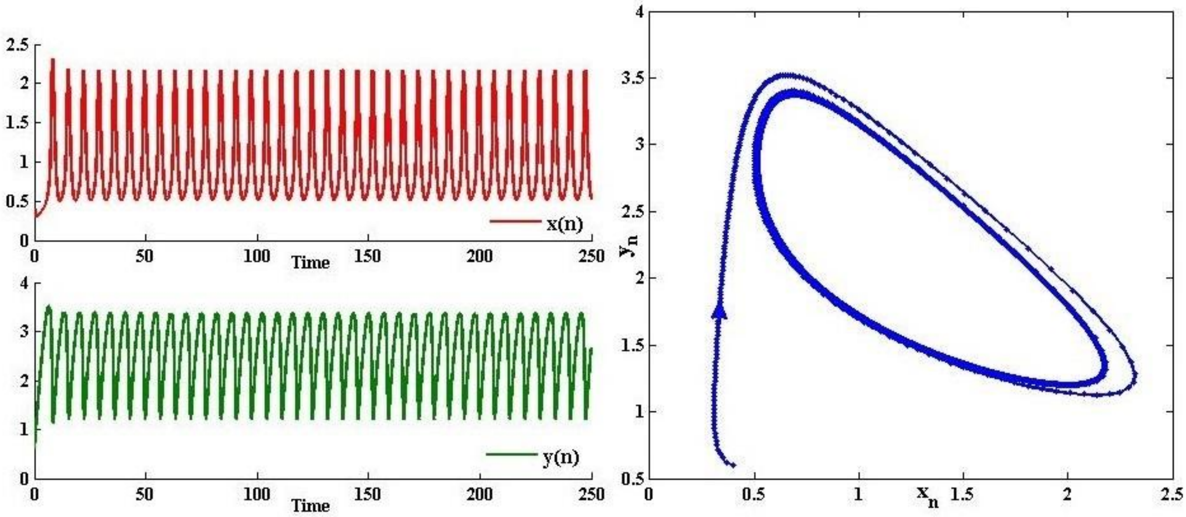

Figure 2: *Time series and Phase portrait are unstable at* E_1 .

6.1.2. Incommensurate Fractional Order $\left(\varepsilon_1 \neq \varepsilon_2\right)$**:**

For the incommensurate nonlinear fractional order system (1), the eigenvalues can be computed by solving the characteristic equation (7) which may be a high degree polynomial function depending on the fractional orders ε_1 and ε_2. If $a = 0.59$ and $b = 1.39$ with the fractional order derivatives $\left(\varepsilon_1, \varepsilon_2\right) = \left(0.91, 0.96\right)$ and the initial conditions $x_0 = 0.3$ and $y_0 = 0.6$. Then the interior (coexistence) fixed point $E_1 = \left(0.59, 2.3559\right)$. Now the characteristic equation of system (1) is given by $\det\left(\operatorname{diag}\left(\left[\lambda^{91}\lambda^{96}\right]\right) - J\right) = 0$, which is simplified as $\lambda^{187} - 0.39\lambda^{96} + 0.3481\lambda^{91} + 0.3481 = 0$. As the characteristic equation is of a very high degree, so it is difficult to find explicit solution of this equation.

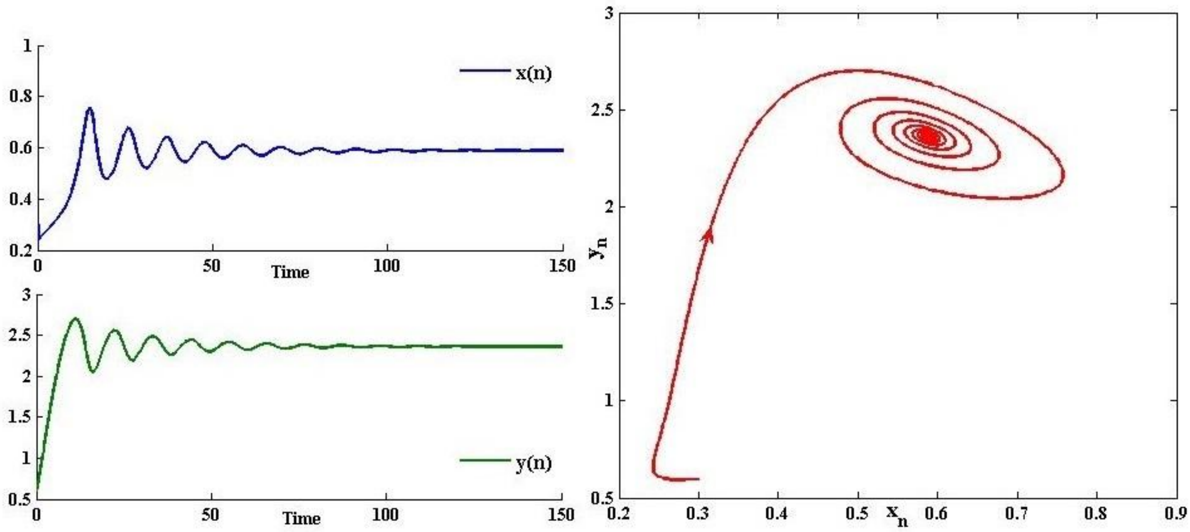

Figure 3: *Time series and Phase portrait are stable at* E_1 .

All the eigenvalues are approximated, which gives the minimum argument: $\min\left\{\left|\arg(\lambda)\right|\right\} = 0.0165$. Since $\gamma = \frac{\pi}{200} = 0.0157$, the fixed point of incommensurate fractional order system (1) is asymptotically stable if $\min\left\{\left|\arg(\lambda)\right|\right\} > \frac{\gamma\pi}{2}$. The stability region with respect to E_1 is shown in Figure 3. For $a = 0.5$, $b = 1.39$ with the fractional order derivatives $\left(\varepsilon_1, \varepsilon_2\right) = \left(0.92, 0.96\right)$ and the initial conditions $x_0 = 0.4$ and $y_0 = 0.6$. We obtain the interior (coexistence) fixed point $E_1 = \left(0.5, 2.78\right)$. Now the characteristic equation of system (1) is given by $\det\left(\text{diag}\left(\left[\lambda^{92}\lambda^{96}\right]\right)-J\right)=0$, which is simplified as $\lambda^{188}-0.39\lambda^{96}+0.25\lambda^{92}+0.25=0$. As the characteristic equation is of a very high degree, it is difficult to find explicit solution of this equation.

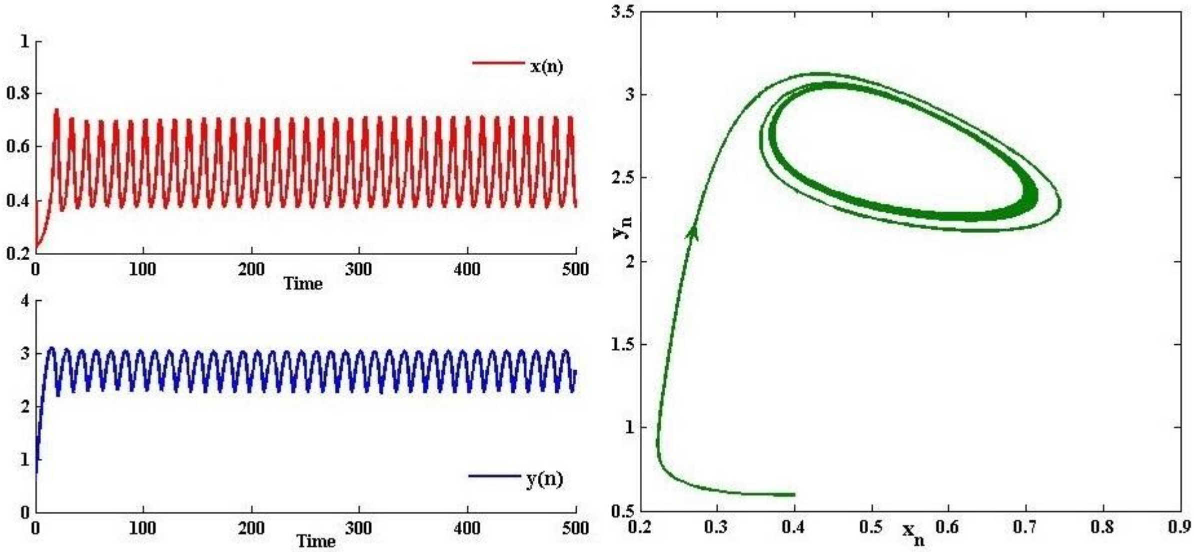

Figure 4: *Time Series and Phase Portrait are unstable at* E_1 .

All the eigen values are approximated, we have $\min\left\{\left|\arg(\lambda)\right|\right\}=0.0153$. Since $\gamma=\frac{\pi}{200}=0.0157$, the incommensurate nonlinear fractional order system (1) will satisfy the non-negativity of the instability condition of its fixed point, $\frac{\gamma\pi}{2}-\min\left\{\left|\arg(\lambda)\right|\right\}\geq 0$. The instability region with respect to E_1 is shown in Figure 4.

6.2. Simulations of Model (8)

The purpose of this section is to present time plots and phase portraits of system (8) to confirm theoretical results. We choose the parameters $\varepsilon=0.245$, $s=0.911$, $a=0.929$, $b=0.927$ with initial values $x_0=0.9$ and $y_0=0.7$. In figure 5, we have the interior fixed point $E_1=(0.9290,0.9978)$ and the eigenvalues are $\lambda_{1,2}=0.4959\pm i\,0.8644$ so that $\left|\lambda_{1,2}\right|=0.9965<1$. In this case E_1 is asymptotically stable (sink) according to Theorem 5, see Figure 5.

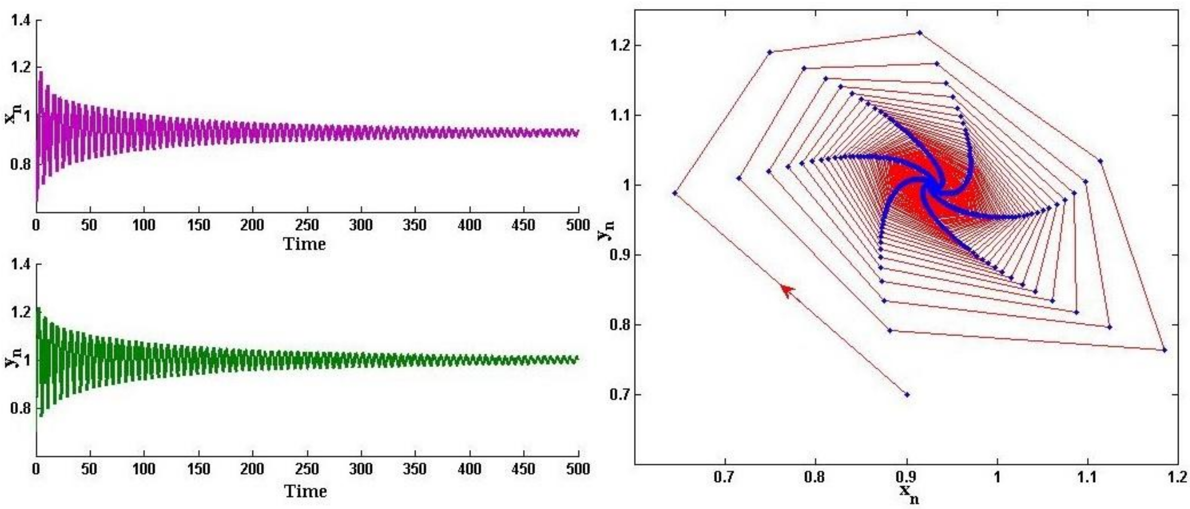

Figure 5: *Time series and Phase portrait are stability at* E_1 .

We consider system (8) with parameters values $\varepsilon=0.209$, $s=0.921$, $a=0.931$, $b=0.937$ and the initial values $x_0=0.9$ and $y_0=0.7$. In Figure 6, we have $E_1=(0.9310,1.0064)$ and the eigenvalues are $\lambda_{1,2}=0.5011\pm i\,0.8657$ so that $\left|\lambda_{1,2}\right|=1.0003>1$. We can easily verify that as the fractional order ε decreases E_1 moves from stability to instability and the trajectory spirals moving inwards but fails to converge to fixed point E_1 and settles down as a limit cycles. In this case E_1 is unstable (saddle), see Figure 6.

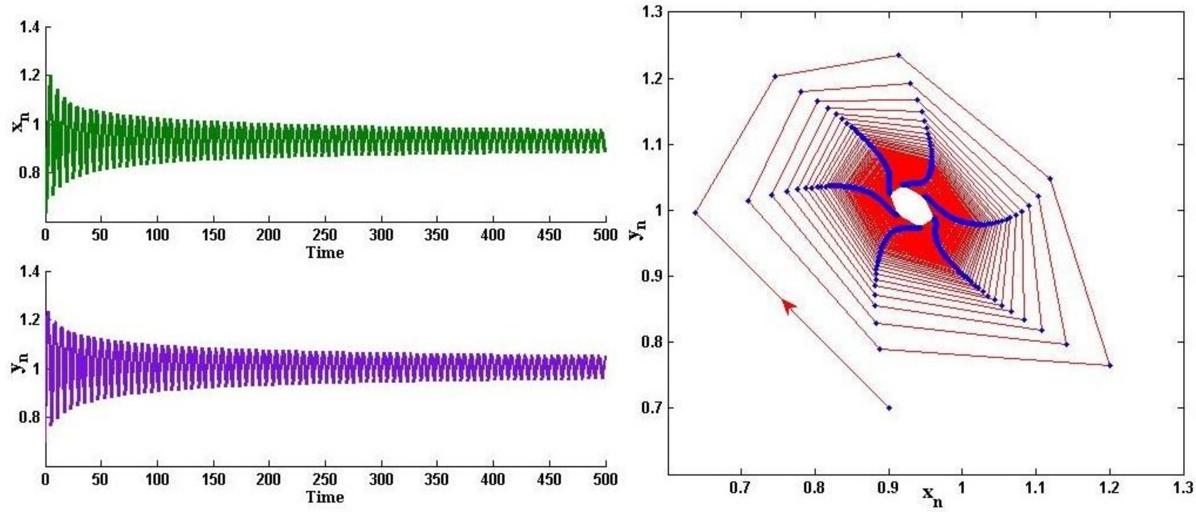

Figure 6: *Time series and Phase portrait are instability at* E_1

7. Bifurcation Analysis

In this section, we investigate the bifurcation parametric conditions for the existence of period doubling (flip) bifurcation and Neimark Sacker bifurcation at the interior fixed point to support the analytical analysis and the complex dynamics of the nonlinear fractional order system (8) with the help of numerical simulations.

7.1. Period Doubling Bifurcation: We can see easily see that for interior fixed point E_1 if a varies in the small neighbourhood of FB_{E_1}, then the period doubling (flip) bifurcation will appear in the system (8) where

$$FB_{E_1} = \left\{ (\varepsilon, s, a, b) : a = a_0 = \sqrt{\frac{2\left(\frac{s^{\varepsilon}}{\Gamma(\varepsilon+1)}(1-b)-2\right)}{\frac{s^{\varepsilon}}{\Gamma(\varepsilon+1)}\left(\frac{s^{\varepsilon}}{\Gamma(\varepsilon+1)}-2\right)}}, \Delta \geq 0,\ a > 1, \alpha, s, b > 0 \right\}.$$

Example 7.1. First, we take $\varepsilon = 0.981$, $s = 0.874$, $b = 0.397$ and $a \in [1.66 - 1.88]$ with initial values $x_0 = 0.1$ and $y_0 = 0.2$ then the system (8) undergoes period doubling(flip) bifurcation as it emerges from the interior fixed point $E_1 = (1.7249, 0.2302)$ as a varies in the small neighbourhood of $a_0 = 1.72494$ and the system dynamics converges to a period- 2 orbit. The corresponding bifurcation diagram is shown in Figure 7. The characteristic polynomial evaluated at this state is given by $\lambda^2 + 1.1605\lambda + 0.1605 = 0$ (10)

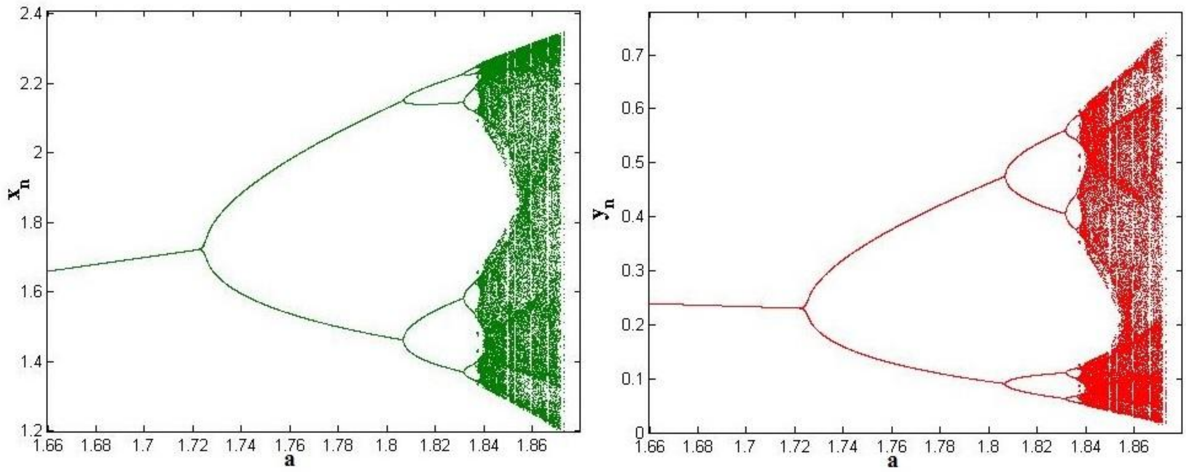

Figure 7: *Flip bifurcation diagrams for system (8) around* E_1 *in (a-x) and (a-y) planes.*

Furthermore, the roots of (10) are $\lambda_1 = -1$ and $\lambda_2 = -0.1605 \neq \pm 1$. Thus we have $(a = 1.72494 \text{ and } \Delta = 0.9035 \geq 0) \in FB_{E_1}$.

From Figure 7, we observe that an interior fixed point E_1 of map (8) is stable for $a < 1.72494$ and loses its stability through a period doubling bifurcation for $a = 1.72494$ and for $a > 1.72494$, we observe periodic doubling cascade in orbits of periods – 2, 4, 8, 16, 32 and non-periodic oscillations appear, that is usually referred to as chaos, see Figure 7.

7.2. Neimark Sacker Bifurcation: In order to discuss a Neimark Sacker bifurcation for the system (8) at the coexistence fixed point E_1, we choose b as bifurcation parameter. From (9) it is observed that $F(\lambda)=0$ has two complex conjugate root with modulus one. The criterion in the terms (iii.b) of Theorem 5 can be presented as follows

$$NS_{E_1}=\left\{(\varepsilon,s,a,b): b=a^2\left(1-\frac{s^\varepsilon}{\Gamma(\varepsilon+1)}\right)+1, \Delta<0,\, b>1, \alpha, s, a>0\right\}$$

then the Neimark – Sacker bifurcation will appear in the system (8) if b varies in the small neighbourhood of NS_{E_1}.

Example 7.2. Let $\varepsilon=0.125$, $s=0.89$, $a=0.927$ and $b\in[0.9-1.35]$ with initial values $x_0=0.9$ and $y_0=0.7$, then the system (8) undergoes a Neimark – Sacker bifurcation it emerges from the interior fixed point $E_1=(0.9270, 1.0356)$ at bifurcation parameter $b=0.96$. The corresponding bifurcation diagram is shown in Figure 8. The characteristic polynomial evaluated at E_1 is given by $\lambda^2-1.0588\lambda+1=0$ (11)

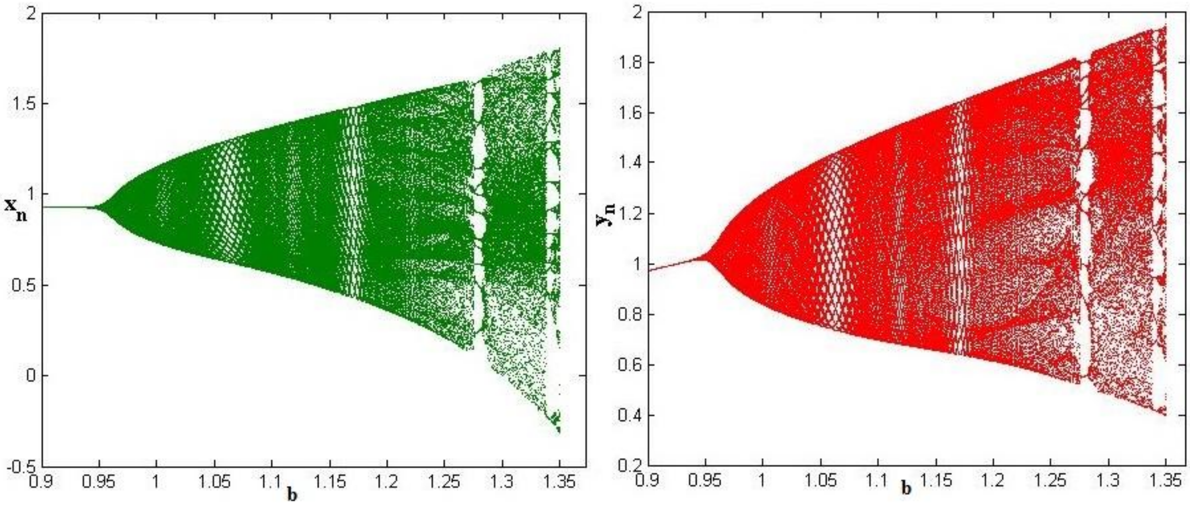

Figure 8: *Neimark – Sacker bifurcation diagrams of system (8) around E_1 in (b-x) and (b-y) planes.*

Furthermore, the roots of (11) are $\lambda_{1,2}=0.5294\pm i\,0.8483$ with $|\lambda_{1,2}|=1$. Thus we have $(b=0.96 \text{ and } \Delta=-2.6286<0)\in NS_{E_1}$. From Figure 8, we observe that coexistence fixed point E_1 of map (8) is stable for $b<0.96$ and loses its stability through a Neimark – Sacker bifurcation for $b=0.96$ and an invariant circle appears for $b>0.96$.

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