International Journal of Technical Innovation in Modern Engineering & Science (IJTIMES)
Impact Factor: 5.22 (SJIF-2017), e-ISSN: 2455-2585
Volume 5, Issue 04, April-2019

ENRESDOWEDNESS OF TYPE – I UNICYCLIC GRAPHS

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Abstract

Let G = (V, E) be a non empty, finite, simple graph. A dominating set of a graph G containing a minimum dominating set of G is called a γ - endowed dominating set of G. If that set is of cardinality k then it is called a k γ – endowed dominating set. k - γ_r enresdowed graph is one in which every restrained dominating set of cardinality k contains a minimum restrained dominating set. A unicyclic graph is a graph consisting of a single cycle. We consider a unicyclic graph of the type , where a set of all vertices of any cycle is attached by a path P_t , $t \geq 2$. In this paper, the enresdowedness property for the unicyclic graphs with exactly one path attached to set of all the vertices of any cycle is found.

Keywords : Enresdowed graphs, Unicyclic graphs.

1. INTRODUCTION

Let G = (V, E) be a non empty, finite, simple graph. A subset D of V(G) is called a dominating set of G if for every v ∈ V – D, there exists u ∈ D such that u and v are adjacent. The minimum cardinality of the dominating set is called the domination number and it is denoted by γ(G)[10]. The restrained dominating set of a graph is a dominating set in which every vertex in V – D is adjacent to some other vertex in V – D. The minimum cardinality of the restrained dominating set is called the restrained domination number and it is denoted by γ_r(G) [3]. A graph is said to be k - γ_r enresdowed graph if every restrained dominating set of cardinality k contains a minimum restrained dominating set [9]. A graph is called unicyclic if it is connected and contains exactly one cycle. A graph is unicyclic if and only if it is connected and has size equal to its order [1]. A family of unicyclic graphs is widely studied by many authors in the theory of domination.

Guo determined the graphs with the first ten maximum spectral radii among all the *n*-vertex unicyclic graphs for n ≥17 [5]. Belardo *et al.* determined the maximum spectral radius of unicyclic graphs with given girth [2] . Yu and Tian gave the first two spectral radii of unicyclic graphs with a given matching number [11]. More results on the spectral radius of unicyclic graphs can be found in [4,5,8].

Unicyclic graphs have applications in different research areas and domains. For example, unicyclic graphs are often used in telecommunications. They allow end-users connected in the same unicyclic component or graph to communicate using the two directions of the cycle. The cycle ensures a certain level of survivability to link failure that occurs on the edges of this cycle (commonly known as a “ring” in telecommunications). The traffic demands between nodes on the same cycle are then fully protected against failures while the other demands can be disrupted.[7]

2. RESULTS ON TYPE – I UNICYCLIC ENRESDOWED GRAPHS

Definition 2.1

Let *k* be a positive integer. A simple, finite, non trivial graph G = (V, E) is called a k - γ_r enresdowed graph if every restrained dominating set of G of cardinality *k* contains a minimum restrained dominating set γ_r of G.[9]

Definition 2.2

A unicyclic graph is a connected graph containing exactly one cycle.

Theorem 2.3

Let G be a unicyclic graph C_nP_t , for $n \geq 3$, $t \geq 2$, where C_n, $n \geq 3$ be the cycle in G with the vertex set $\{ v_i \}$, $1 \leq i \leq n$ and P_t be a path of G with the vertex set $\{ u_{i1}$, $u_{i2}, \ldots, u_{ir_i}, \ldots, u_{im_i} \}$, $1 \leq i \leq n$, $1 \leq r_i \leq m_i$, where the initial vertex u_{i1}, $1 \leq i \leq n$ of a single path P_t, $t \geq 2$ is attached to every vertex v_i of the cycle such that $u_{i1} = v_i$, then for any C_nP_t ,

(1) If $P_t = P_{2t_1+j}$, for $t_1 = 1$ and j = 0,3,6,…. and for C_n , $n \geq 3$, then G is k - γ_r enresdowed for any *k*, where $\gamma_r \leq k \leq n$, except for *k* = n – 1.

(2) If $P_t = P_{3t_1+j}$, for $t_1 = 1$ and j = 0,3,6,…. and for C_n , $n \geq 3$ then G is k - γ_r enresdowed for any *k*, where $\gamma_r \leq k \leq n$, except for *k* = n – 1.

(3) If $P_t = P_{3t_1+1}$, for $t_1 \geq 1$ and for C_n, $n \geq 3$, then

3(a) If the γ_r set contains the entire vertex set of C_n, then G is k - γ_r enresdowed for any *k*, where $\gamma_r + l \leq k \leq n$, for even $l \geq 0$, except for *k* = n – 1.

3(b) If the γ_r set contains some of the vertices of the cycle C_n , then G is k - γ_r enresdowed for any *k*, where $\gamma_r \leq k \leq n$, except for *k* = n – 1.

Proof

Given G = C_nP_t , for $n \geq 3$, $t \geq 2$ is a unicyclic graph which consist of a cycle C_n, $n \geq 3$ and a path P_t , $t \geq 2$. The vertex set of C_n is $V(C_n) = \{v_1, v_2, v_3, \ldots, v_i, \ldots, v_n\}$, $1 \leq i \leq n$ and the vertex set of the path P_t be $V(P_t) = \{ u_{11}, u_{12}, u_{13}, \ldots, u_{1r_1}, \ldots, u_{1m_1}, u_{21}, u_{22}, u_{23}, \ldots, u_{2r_2}, \ldots, u_{2m_2}, u_{31}, u_{32}, u_{33}, \ldots, u_{3r_3}, \ldots, u_{3m_3}, \ldots, u_{i1}, u_{i2}, \ldots, u_{ir_i}, \ldots, u_{im_i}, \ldots, u_{n1}, u_{n2}, u_{n3}, \ldots, u_{nr_n}, \ldots, u_{nm_n}\}$, $1 \leq i \leq n$, $1 \leq r_i \leq m_i$. Thus the vertex set of P_t is the union of the vertex sets of the paths $P_{t_1}, P_{t_2}, \ldots, P_{t_i}, \ldots, P_{t_n}$, $1 \leq i \leq n$ where the set of vertices $\{u_{11}, u_{12}, u_{13}, \ldots, u_{1r_1}, \ldots, u_{1m_1}\}$ of the path P_{t_1} is attached to the vertex v_1 of C_n , $n \geq 3$ such that the vertex $v_1 = u_{11}$, similarly the set of vertices $\{u_{21}, u_{22}, u_{23}, \ldots, u_{2r_2}, \ldots, u_{2m_2}\}$ of the path P_{t_2} is attached to the vertex v_2 of C_n, $n \geq 3$ such that the vertex $v_2 = u_{21}$,without loss of generality, consider the set of vertices $\{u_{i1}, u_{i2}, \ldots, u_{ir_i}, \ldots, u_{im_i}\}$ are attached to the vertex v_i of C_n, $n \geq 3$, $1 \leq i \leq n$ such that the vertex $v_i = u_{i1}$, $1 \leq i \leq n$. Similarly the set of vertices $\{ u_{n1-1}, u_{n2-1}, u_{n3-1}, \ldots, u_{nr_n-1}, \ldots, u_{nm_n-1}\}$ are attached to the vertex v_{n-1} of C_n, $n \geq 3$, such that the vertex $v_{n-1} = u_{n1-1}$. Finally the set of vertices $\{u_{n1}, u_{n2}, u_{n3}, \ldots, u_{nr_n}, \ldots, u_{nm_n}\}$are attached to the vertex v_n of C_n, $n \geq 3$ such that the vertex $v_n = u_{n1}$. Thus the vertex set of G is the union of the vertex set of the path P_t , $t \geq 2$ and the cycle C_n, $n \geq 3$. Hence the following cases exists

Case (i) Consider any graph $G_1 = C_nP_t$, where C_n , $n \geq 3$ be the cycle and P_t , $t \geq 2$ be the path of G_1. Let $P_t = P_{2t_1+j}$, for $t_1 = 1$ and j = 0,3,6,9,…, and then the set of all vertices $\{v_i\}$, $1 \leq i \leq n$ of the cycle C_n ,$n \geq 3$ is attached with the paths of the type $P_2, P_5, P_8, \ldots$ then there exists the following subcases.

Subcase (i)(a) Consider the graph $G_{11} = C_nP_2$, for $n \geq 3$. Without loss of generality, assume that n = 3 for the cycle C_n, then the graph $G_{11} = C_3P_2$ is obtained. Let D_1 be the γ_r set of G_{11}. The vertex set of the cycle C_3 is $\{v_1, v_2, v_3\}$ and the path P_2 is $\{u_{11}, u_{12}, u_{21}, u_{22}, u_{31}, u_{32}\}$, where the vertex $v_1 = u_{11}$ is adjacent to u_{12} and the vertices $v_2 = u_{21}$ and $v_3 = u_{31}$ where the vertices u_{21} and u_{31} is adjacent to the vertices u_{22} and u_{32}. By choosing the set of pendant vertices $\{u_{12}, u_{22}, u_{32}\}$ for the γ_r set D_1, the remaining vertices v_1, v_2, v_3 of the cycle C_3 is dominated and they are adjacent in $V - D_1$. Thus the set $D_1 = \{u_{12}, u_{22}, u_{32}\}$ forms the minimum restrained dominating set of G with cardinality $k_1 = \gamma_r$. Hence G_{11} is k_1 - γ_r enresdowed for any $k_1 = \gamma_r$. Consider any set D_2 of cardinality $k_2 = \gamma_r + 1$, then $D_2 = D_1 \cup \{v_i\}, 1 \leq i \leq 3$, where D_2 forms a restrained dominating set of cardinality γ_r +1, and it contains the γ_r set D_1. Hence G_{11} is k_2 - γ_r enresdowed. Similarly consider a set D_3 of cardinality $k_3 = \gamma_r + 2 = n - 1$. Thus G_{11} is not k_3 - γ_r enresdowed. Finally consider the set D_4 of cardinality $k_4 = \gamma_r + 3 = n$, where $D_4 = D_1 \cup \{v_1, v_2, v_3\}$ form the restrained dominating set with cardinality $k_4 = n$. Thus G_{11} is k_4 - γ_r enresdowed for any $k_4 = n$. In this case $G_{11} = C_nP_2$, for $n \geq 3$ is k - γ_r enresdowed for any k, where $\gamma_r \leq k \leq n$ except for $k = n - 1$.

Subcase (i)(b) Consider the graph $G_{12} = C_nP_5$, for $n \geq 3$. Without loss of generality, assume that n = 4 for the cycle C_n, then the graph $G_{12} = C_4P_5$ is obtained. The vertex set of the cycle C_4 is $\{v_1, v_2, v_3, v_4\}$ and the path P_5 is $\{u_{11}, u_{12}, u_{13}, u_{14}, u_{15}, u_{21}, u_{22}, u_{23}, u_{24}, u_{25}, u_{31}, u_{32}, u_{33}, u_{34}, u_{35}, u_{41}, u_{42}, u_{43}, u_{44}, u_{45}\}$, where the set of vertices $\{u_{1r_1}, u_{2r_2}, u_{3r_3}, u_{4r_4}\}, 1 \leq r_1, r_2, r_3, r_4, r_5 \leq 5$ forms the vertex set of paths which is adjacent to the set of vertices $\{v_1, v_2, v_3, v_4\}$. Choose the set of pendant vertices $\{u_{15}, u_{25}, u_{35}, u_{45}\}$ for the γ_r set D_5 then the vertices $\{u_{14}, u_{24}, u_{34}, u_{44}\}$ are dominated, similarly choose the set of vertices $\{u_{12}, u_{22}, u_{32}, u_{42}\}$ for the γ_r set D_5, then the set of vertices $\{u_{13}, u_{23}, u_{33}, u_{43}\}$ and $\{u_{11}, u_{21}, u_{31}, u_{41}\}$ are dominated. Thus the set $D_5 = \{u_{15}, u_{25}, u_{35}, u_{45}, u_{12}, u_{22}, u_{32}, u_{42}\}$ and $V - D_5 = \{u_{14}, u_{24}, u_{34}, u_{44}, u_{13}, u_{23}, u_{33}, u_{43}, u_{11}, u_{21}, u_{31}, u_{41}\}$ forms a minimum restrained dominating set of G_{12} of cardinality $k_5 = \gamma_r$. Hence G_{12} is k_5 - γ_r enresdowed for any $k_5 = \gamma_r$. Similarly consider the set D_6 of cardinality $k_6 = \gamma_r + 1$, where there exists two subcases

Subcase (i)(b_1) Consider the set $D_{61} = D_5 \cup \{u_{pr_i}\}$, $1 \leq p \leq 4$, $r_i = 3,4$ where the vertex $\{u_{pr_i}\}$, $1 \leq p \leq 4$, $r_i = 3, 4$ belong to the path of G. Without loss of generality, assume that the vertex $u_{pr_i} = u_{14}$ for p = 1, $r_i = 4$. Consider the set D_{61}, where $D_{61} = D_5 \cup \{u_{14}\}$, then the vertex u_{13} in $V - D_{61}$ is an isolate, thus the set D_{61} which is of cardinality $k_{61} = \gamma_r + 1$ is not a restrained dominating set. Hence G_{12} is not k_{61} - γ_r enresdowed.

Subcase (i)(b_2) Consider the set $D_{62} = D_5 \cup \{u_{pr_i}\}$, $1 \leq p \leq 4$, $r_i = 1$, where the vertex $\{u_{pr_i}\}$, $1 \leq p \leq 4$, $r_i = 1$ belong to the cycle C_4, of G. Assume that p = 2, then the vertex $u_{pr_i} = u_{21}$. Thus the set $D_{62} = D_5 \cup \{u_{21}\}$, and $V - D_{62} = \{u_{14}, u_{24}, u_{34}, u_{44}, u_{13}, u_{23}, u_{33}, u_{43}, u_{11}, u_{31}, u_{41}\}$ forms the restrained dominating set of G_{12} with cardinality $k_{62} = \gamma_r + 1$, containing the minimum restrained dominating set D_5 of G_{12}. Hence G_{12} is k_{62} - γ_r enresdowed.

Consider a set D_7 of cardinality $k_7 = \gamma_r + 2$, where there exists three subcases.

Subcase (i)(b_3) Consider the set $D_{71} = D_5 \cup \{u_{p_1r_{i1}}, u_{p_2r_{i2}}\}$, $1 \leq p_1, p_2 \leq 4$, and $r_{i1}, r_{i2} = 3$, 4, such that $p_1 = p_2$, $r_{i1} \neq r_{i2}$ where $r_{i1}, r_{i2} \neq 1$. The set D_{71} is of cardinality $k_{71} = \gamma_r + 2$. Without loss of generality, assume that the vertices $u_{p_1r_{i1}} = u_{13}$ and $u_{p_2r_{i2}} = u_{14}$. Then the set D_{71} is $D_{71} = \{u_{15}, u_{25}, u_{35}, u_{45}, u_{12}, u_{22}, u_{32}, u_{42}, u_{13}, u_{14}\}$ and $V - D_{71} = \{u_{24}, u_{34}, u_{44}, u_{23}, u_{33}, u_{43}, u_{11}, u_{21}, u_{31}, u_{41}\}$. Thus the set D_{71} forms the restrained dominating set of cardinality $k_{71} = \gamma_r + 2$, containing the γ_r set D_5 of G_{12}. Hence G_{12} is k_{71} - γ_r enresdowed for any $k_{71} = \gamma_r + 2$.

Subcase (i)(b_4) Consider the set $D_{72} = D_5 \cup \{u_{p_1r_{i1}}, u_{p_2r_{i2}}\}$, $1 \le p_1, p_2 \le 4$, and $r_{i1} = r_{i2} = 1$, such that $p_1 \ne p_2$ and the vertices $u_{p_1r_{i1}}$ and $u_{p_2r_{i2}}$ are adjacent. The cardinality of the set D_{72} is $k_{72} = \gamma_r + 2$, without loss of generality, assume that the vertex $u_{p_1r_{i1}} = u_{11}$ and $u_{p_2r_{i2}} = u_{21}$, then the set D_{72}, where $D_{72} = D_5 \cup \{u_{11}, u_{21}\}$, where $V - D_{72} = \{u_{24}, u_{34}, u_{44}, u_{23}, u_{33}, u_{43}, u_{31}, u_{41}, u_{13}, u_{14}\}$ forms the restrained dominating set of cardinality $k_{72} = \gamma_r + 2$ containing the γ_r set D_5 of G_{12}. Hence G_{12} is k_{72} - γ_r enresdowed.

Subcase (i)(b_5) Consider the set $D_{73} = D_5 \cup \{u_{p_1r_{i1}}, u_{p_2r_{i2}}\}$, $1 \le p_1, p_2 \le 4$, and either $r_{i1} = 1$ or $r_{i2} = 1$. The set D_{73} is of cardinality $k_{73} = \gamma_r + 2$. Without loss of generality, assume that $r_{i1} = 1$ and $r_{i2} \ne 1$ and the vertices $u_{p_1r_{i1}} = u_{11}$ and $u_{p_2r_{i2}} = u_{23}$. Then the set $D_{73} = \{u_{15}, u_{25}, u_{35}, u_{45}, u_{12}, u_{22}, u_{32}, u_{42}, u_{11}, u_{23}\}$ and $V - D_{73} = \{u_{14}, u_{24}, u_{34}, u_{44}, u_{13}, u_{33}, u_{43}, u_{21}, u_{31}, u_{41}\}$. In the set, $V - D_{73}$ the vertex u_{24} is an isolate, thus the set D_{73} is not a restrained dominating set. Hence G_{12} is not k_{73} - γ_r enresdowed.

Proceeding similarly, consider the set D_8 of cardinality $k_8 = n - 1$, where $D_8 = D_5 \cup \{u_{14}, u_{24}, u_{34}, u_{44}, u_{13}, u_{23}, u_{33}, u_{43}, u_{11}, u_{21}, u_{31}\}$ and $V - D_8 = \{u_{41}\}$. Thus D_8 is not a restrained dominating set of G_{12}. Hence G_{12} is not k_8 - γ_r enresdowed for any $k_8 = n - 1$. Finally, consider the set D_9, where $D_9 = D_5 \cup (V - D_5)$ is of cardinality $k_9 = n$, which contains the γ_r set D_5. Thus D_9 is a restrained dominating set containing the minimum restrained dominating set D_5 of G_{12}. Hence G_{12} is k_9 - γ_r enresdowed for any $k_9 = n$.

Subcase (i)(c) In general, consider the graph $G_{1_{n_1}} = C_n P_{n_i}$, for $n \ge 3$, $i = 2,5,8,\ldots$. The vertex set of the cycle C_n is $\{v_1, v_2, v_3, \ldots, v_i, \ldots, v_n\}$ and the path be $\{u_{11}, u_{12}, u_{13}, \ldots, u_{1r_1}, \ldots, u_{1m_1}, u_{21}, u_{22}, u_{23}, \ldots, u_{2r_2}, \ldots, u_{2m_2}, u_{31}, u_{32}, u_{33}, \ldots, u_{3r_3}, \ldots, u_{3m_3}, \ldots, u_{i1}, u_{i2}, \ldots, u_{ir_i}, \ldots, u_{im_i}, \ldots, u_{n1}, u_{n2}, u_{n3}, \ldots, u_{nr_n}, \ldots, u_{nm_n}\}$, $1 \le i \le n$, $1 \le r_i \le m_i$, where the vertex $u_{11} = v_1$, $u_{21} = v_2, \ldots, u_{i1} = v_i, \ldots, u_{n1} = v_n$. Choose the vertex u_{12} for the γ_r set D_{10}, then the vertices u_{11}, u_{13} is dominated, also choose the vertex u_{15}, then the vertices u_{14} and u_{16} are dominated, where the vertices u_{13} and u_{14} are adjacent in $V - D_{10}$. Proceeding similarly, choose the vertex u_{1m_1} for the γ_r set D_{10}, thus the set of vertices $\{u_{12}, u_{15}, u_{18}, \ldots, u_{1m_1}\}$ belongs to the γ_r set D_{10}, from the path P_{t_1} attached to the vertex v_1, similarly the set of vertices $\{u_{22}, u_{25}, u_{28}, \ldots, u_{2m_2}\}$, $\{u_{32}, u_{35}, u_{38}, \ldots, u_{3m_3}\}, \ldots, \{u_{i2}, u_{i5}, u_{i8}, \ldots, u_{im_i}\}, \ldots, \{u_{n2-1}, u_{n5-1}, \ldots, u_{nm_n-1}\}$, $\{u_{n2}, u_{n5}, u_{n8}, \ldots, u_{nm_n}\}$, belongs to the γ_r set D_{10} from the paths $P_{t_2}, P_{t_3}, \ldots, P_{t_i}, \ldots, P_{t_{n-1}}, P_{t_n}$ which are attached to the vertices $v_2, v_3, \ldots, v_i, \ldots, v_{n-1}, v_n$ of the cycle C_n. Then the γ_r set D_{10} is of cardinality $k_{10} = \gamma_r$ where the γ_r set $D_{10} = \{u_{12}, u_{15}, u_{18}, \ldots, u_{1m_1}, u_{22}, u_{25}, u_{28}, \ldots, u_{2m_2}, u_{32}, u_{35}, u_{38}, \ldots, u_{3m_3}, \ldots, u_{i2}, u_{i5}, u_{i8}, \ldots, u_{im_i}, \ldots, u_{n2-1}, u_{n5-1}, \ldots, u_{nm_n-1}, u_{n2}, u_{n5}, u_{n8}, \ldots, u_{nm_n}\}$ and $V - D_{10} = \{u_{11}, u_{13}, u_{14}, \ldots, u_{1m_1-1}, u_{21}, u_{23}, u_{24}, \ldots, u_{2m_2-1}, u_{31}, u_{33}, u_{34}, \ldots, u_{3m_3-1}, \ldots, u_{i1}, u_{i3}, u_{i4}, \ldots, u_{im_i-1}, \ldots, u_{n1-1}, u_{n3-1}, u_{n4-1}, \ldots, u_{nm_n-2}, u_{n1}, u_{n3}, u_{n4}, \ldots, u_{nm_n-1}\}$. Thus the set D_{10} forms the minimum restrained dominating set of $G_{1_{n_1}}$ of cardinality $k_{10} = \gamma_r$. Hence $G_{1_{n_1}}$ is k_{10} - γ_r enresdowed.

Consider the set D_{11} of cardinality $k_{11} = \gamma_r + 1$, where there exists two subcases.

Subcase (i)(c_1) Consider the set $D_{11,1} = D_{10} \cup \{u_{pr_i}\}$, $1 \le p \le n$, $r_i \ne q + 1$, for $q = 0,1,4,7,10,\ldots$ where the vertex $\{u_{pr_i}\}$, belong to the path P_{t_i} $1 \le i \le n$, of $G_{1_{n_1}}$. Since the vertex $\{u_{pr_i}\}$ is adjacent only to the vertex $u_{p(r_i+1)}$ in $V - D_{10}$, thus the vertex $u_{p(r_i+1)}$ is an isolate in $V - D_{11,1}$. Therefore $D_{11,1}$ is not a restrained dominating set. Hence $G_{1_{n_1}}$ is not $k_{11,1}$ - γ_r enresdowed for any $k_{11,1} = \gamma_r + 1$.

Subcase (i)(c_2) Consider the set $D_{11,2}$ =$D_{10} \cup \{u_{pr_i}\}$, $1 \leq p \leq n$, $r_i = 1$, then the vertex $\{u_{pr_i}\}$ belong to the set$\{u_{11}, u_{21}, u_{31},\ldots, u_{i1},\ldots, u_{n1}\}$, $1 \leq i \leq n$ of the cycle C_n, for $n \geq 3$ of $G_{1_{n_1}}$. Without loss of generality, choose $u_{pr_i} = u_{i1}$, then the set $V - D_{11,2}$ has no isolates since the vertices in the set $\{u_{11}, u_{21}, u_{31},\ldots, u_{i1},\ldots\ldots u_{n1}\}$ are adjacent. Thus $D_{11,2}$ forms the restrained dominating set containing the γ_r set D_{10}, where $D_{11,2}$ is of cardinality $k_{11,2} = \gamma_r +1$, Hence $G_{1_{n_1}}$is $k_{11,2}$ - γ_r enresdowed for any $k_{11,2} = \gamma_r +1$.

Consider a set D_{12} of cardinality $k_{12} = \gamma_r + 2$, then there exists the following subcases

Subcase (i)(c_3) Consider the set $D_{12,1} = D_{10} \cup \{u_{p_1r_{i1}}, u_{p_2r_{i2}}\}$, $1 \leq p_1, p_2 \leq n$, $r_i \neq q + 1$ for $q = 0,1,4,7,10,\ldots$, such that $p_1= p_2$ and $r_{i1} \neq r_{i2}$. The set $D_{12,1}$ is of cardinality $k_{12,1} = \gamma_r +2$. Since $r_{i1}, r_{i2} \neq 1$, these vertices belong to the path P_{t_i} $1 \leq i \leq n$ of $G_{1_{n_1}}$. Choose the vertices $u_{p_1r_{i1}}$, $u_{p_2r_{i2}}$ in such a way that they are adjacent in $V - D_{10}$. Therefore there exists no isolates in $V - D_{12,1}$. Hence $D_{12,1}$ forms the restrained dominating set of cardinality $k_{12,1} = \gamma_r +2$, containing the γ_r set D_{10}. Thus $G_{1_{n_1}}$ is $k_{12,1}$ - γ_r enresdowed.

Subcase (i)(c_4) Consider the set $D_{12,2} = D_{10} \cup \{u_{p_1r_{i1}}, u_{p_2r_{i2}}\}$, $1 \leq p_1, p_2 \leq n$, and $r_{i1} = r_{i2} = 1$, such that $p_1 \neq p_2$, where these vertices $u_{p_1r_{i1}}$, $u_{p_2r_{i2}}$ belong to the vertex set of cycle C_n , $n \geq 3$. The cardinality of the set $D_{12,2}$ is $k_{12,2} = \gamma_r + 2$. Choose the vertices $u_{p_1r_{i1}}$, $u_{p_2r_{i2}}$ in such a way that they are adjacent in $V - D_{10}$. Similarly in this case, there exists no isolates in $V - D_{12,2}$. Therefore $D_{12,2}$ forms the restrained dominating set containing the γ_r set D_{10}. Thus $G_{1_{n_1}}$is $k_{12,2}$ - γ_r enresdowed, for any $k_{12,2} = \gamma_r + 2$.

Subcase (i)(c_5) Consider the set $D_{12,3} = D_{10} \cup \{u_{p_1r_{i1}}, u_{p_2r_{i2}}\}$, $1 \leq p_1, p_2 \leq n$ and either $r_{i1} = 1$ or $r_{i2} = 1$, without loss of generality, assume $r_{i1} = 1$, and $r_{i2} \neq 1$, then the vertex $u_{p_1r_{i1}}$ belongs to the cycle C_n , $n \geq 3$ and the vertex $u_{p_2r_{i2}}$ belongs to the path P_{t_i} $1 \leq i \leq n$ of $G_{1_{n_1}}$, where the vertex $u_{p_1r_{i1}}$ is not adjacent with $u_{p_2r_{i2}}$, also the vertex $u_{p_2r_{i2}}$ is adjacent only to the vertex $u_{p_2r_{(i2+1)}}$ in $V - D_{10}$. Thus by choosing the vertex $u_{p_2r_{i2}}$ for the set $D_{12,3}$, the vertex $u_{p_2r_{(i2+1)}}$ is an isolate in $V - D_{12,3}$. Therefore the set $D_{12,3}$ is not a restrained dominating set. Hence $G_{1_{n_1}}$ is not $k_{12,3}$ - γ_r enresdowed, for any $k_{12,3} = \gamma_r + 2$.

Proceeding similarly, consider the set D_{13} of cardinality $k_{13} = n - 1$, where$D_{13} = D_{10} \cup \{u_{13}, u_{14},\ldots, u_{1m_1-1}, u_{21}, u_{23}, u_{24},\ldots, u_{2m_2-1}, u_{31}, u_{33}, u_{34},\ldots, u_{3m_3-1},\ldots, u_{i1}, u_{i3}, u_{i4},\ldots, u_{im_i-1},\ldots, u_{n1-1}, u_{n3-1}, u_{n4-1},\ldots, u_{nm_n-2}, u_{n1}, u_{n3}, u_{n4},\ldots, u_{nm_n-1}\}$ and $V - D_{13} = \{u_{11}\}$. Thus the set D_{13} is not a restrained dominating set of $G_{1_{n_1}}$. Hence $G_{1_{n_1}}$is not k_{13} - γ_r enresdowed. Finally consider the set D_{14}, where $D_{14} = D_{13} \cup \{u_{11}\}$, where D_{14} forms a restrained dominating set of cardinality $k_{14} = n$, where it contains the minimum restrained dominating set D_{10} of $G_{1_{n_1}}$. Thus $G_{1_{n_1}}$ is k_{14} - γ_r enresdowed.

Case (ii) Consider any graph $G_2 = C_nP_t$, where C_n, $n \geq 3$, be the cycle of G_2 and P_t be a path of G_2. Let $P_t = P_{3t_1+j}$, for $t_1 = 1$ and $j = 0,3,6,9,\ldots$ then the cycle C_n is attached with the paths of the type P_3, P_6, P_9, P_{12},…, then there exists the following subcases

Subcase (ii)(a) Consider a graph $G_{21} = C_n P_t$, where $n = 4$, $t = 3$. In particular, $G_{21} = C_4 P_3$, the vertex set of the cycle C_4 be $\{v_1, v_2, v_3, v_4\}$ and the vertices of the path P_3 which are adjacent to $\{v_1, v_2, v_3, v_4\}$ be $\{u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, u_{23}, u_{31}, u_{32}, u_{33}, u_{41}, u_{42}, u_{43}\}$ where the vertices u_{11}, u_{12}, u_{13} belong to the path P_{t_1} and the path P_{t_1} is attached to the vertex v_1 and u_{21}, u_{22}, u_{23} are the vertices of the path P_{t_2} which are attached to the vertex v_2. Similarly the vertices u_{31}, u_{32}, u_{33} belong to the path P_{t_3} where the path P_{t_3} is attached to the vertex v_3 and u_{41}, u_{42},

u_{43} belong to the path P_{t_4} where the path P_{t_4} is attached to the vertex v_4 of C_n, such that the vertex $v_1 = u_{11}$, $v_2 = u_{21}$, $v_3 = u_{31}$, $v_4 = u_{41}$.

Choose the vertices u_{11}, u_{12}, u_{13} for the γ_r set D_{15}, then the vertices u_{21} and u_{41} which are adjacent to u_{11} are dominated, similarly choose the vertices u_{23} and u_{43} for the γ_r set D_{15}, then the vertices u_{22} and u_{42} are dominated , choose the vertices u_{33} and u_{32} which dominates the vertex v_3, then the vertices $\{u_{21}, u_{22}, u_{31}, u_{41}, u_{42}\}$ are adjacent. Thus the set D_{15} is, $D_{15} = \{u_{11}, u_{12}, u_{13}, u_{23}, u_{32}, u_{33}, u_{43}\}$ and the set $V - D_{15} = \{u_{21}, u_{22}, u_{31}, u_{41}, u_{42}\}$ forms the minimum restrained dominating set of G_{21} with cardinality $k_{15} = \gamma_r$.

Consider a set D_{16} of cardinality $k_{16} = \gamma_r + 1$, then $D_{16} = D_{15} \cup \{u_{pr_i}\}$, where $2 \leq p \leq 4$, $r_i = 1, 2$, such that $u_{pr_i} \neq u_{32}$, then the following subcases exists

Subcase (ii)(a_1) Consider the set $D_{16,1} = D_{15} \cup \{u_{22}\}$, where u_{22} belong to the path P_{t_2} of G_{21}, then the set $V - D_{16,1} = \{u_{21}, u_{31}, u_{41}, u_{42}\}$, where there exists no isolate vertex in $V - D_{16,1}$. Thus the set $D_{16,1}$ is a restrained dominating set of cardinality $k_{16,1} = \gamma_r + 1$, which contains the γ_r set D_{15} of G_{21}. Hence G_{21} is $k_{16,1}$- γ_r enresdowed for any $k_{16,1} = \gamma_r + 1$.

Subcase (ii)(a_2) Consider the set $D_{16,2} = D_{15} \cup \{u_{41}\}$, where the vertex u_{41} belong to the cycle C_n of G_{21}, then the set $V - D_{16,2} = \{u_{21}, u_{22}, u_{31}, u_{42}\}$, in which the vertex u_{42} is an isolate. Thus the set $D_{16,2}$ is not a restrained dominating set. Hence G_{21} is not $k_{16,2}$- γ_r enresdowed for any $k_{16,2} = \gamma_r + 1$.

Consider the set D_{17} of cardinality $k_{17} = \gamma_r + 2$, where $D_{17} = D_{15} \cup \{u_{p_1 r_{i1}}, u_{p_2 r_{i2}}\}$, $2 \leq p_1, p_2 \leq 4$, and $r_{i1}, r_{i2} = 1, 2$ such that $u_{p_1 r_{i1}}, u_{p_2 r_{i2}} \neq u_{32}$, then the following subcases exists

Subcase (ii)(a_3) Consider the set $D_{17,1} = D_{15} \cup \{u_{21}, u_{22}\}$ and the set $V - D_{17,1}$ is $V - D_{17,1} = \{u_{31}, u_{41}, u_{42}\}$, since there exists no isolate vertex in $V - D_{17,1}$, the set $D_{17,1}$ forms the restrained dominating set of cardinality $k_{17,1} = \gamma_r + 2$, containing the minimum restrained dominating set D_{15} of G_{21}. Hence G_{21} is $k_{17,1}$ - γ_r enresdowed.

Subcase (ii)(a_4) Consider the set $D_{17,2} = D_{15} \cup \{u_{31}, u_{41}\}$, then the set $V - D_{17,2}$ is $V - D_{17,2} = \{u_{21}, u_{22}, u_{42}\}$, thus the vertex u_{42} is not adjacent with any vertex in $V - D_{17,2}$, so that the vertex u_{42} is an isolate in $V - D_{17,2}$. Thus the set $D_{17,2}$ is not a restrained dominating set of G_{21}. The cardinality of the set $D_{17,2}$ is $k_{17,2} = \gamma_r + 2$. Hence G_{21} is not $k_{17,2}$ - γ_r enresdowed.

Proceeding similarly, consider the set D_{18} of cardinality $k_{18} = n - 1$, where the set D_{18} is, $D_{18} = D_{15} \cup \{u_{21}, u_{22}, u_{31}, u_{41}\}$ and the set $V - D_{18}$ is $V - D_{18} = \{u_{42}\}$. Thus the set D_{18} is not restrained dominating set and G_{21} is not k_{18} - γ_r enresdowed. Finally consider the set D_{19} of cardinality $k_{19} = n$, where $D_{19} = D_{18} \cup \{u_{42}\}$, forms the restrained dominating set of cardinality n, containing the γ_r set D_{15}. Hence G_{21} is k_{19} - γ_r enresdowed.

Subcase (ii)(b) Consider the graph $G_{22} = C_n P_3$, $n \geq 3$. Without loss of generality, assume that n = 6 for the cycle C_n. Then the graph $G_{22} = C_6 P_3$ is obtained. The vertex set of the cycle C_6 is $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ and the path P_3 is $\{u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, u_{23}, u_{31}, u_{32}, u_{33}, u_{41}, u_{42}, u_{43}, u_{51}, u_{52}, u_{53}, u_{61}, u_{62}, u_{63}\}$, where the set of vertices u_{11}, u_{12}, u_{13} which belong to the path P_{t_1} are attached to the vertex $v_1 = u_{11}$, similarly the vertices u_{21}, u_{22}, u_{23} which belong to the path P_{t_2} are attached to the vertex $v_2 = u_{21}$, where the vertices u_{31}, u_{32}, u_{33}, which belong to the path P_{t_3} are attached to the vertex $v_3 = u_{31}$ and the vertices u_{41}, u_{42}, u_{43} which belong to the path P_{t_4} are attached to the vertex $v_4 = u_{41}$. The vertices u_{51}, u_{52}, u_{53} which belong to the path P_{t_5} are attached to the vertex $v_5 = u_{51}$ and finally the vertices u_{61}, u_{62}, u_{63} which belong to the path P_{t_6} are attached to the vertex $v_6 = u_{61}$. Choose the set of all pendant vertices
$\{u_{13}, u_{23}, u_{33}, u_{43}, u_{53}, u_{63}\}$ for the γ_r set D_{20} of G_{22} so the vertices

{ u_{12} , $u_{22}, u_{32}, u_{42}, u_{52}, u_{62}$ } are dominated.

Now to choose the vertices from the cycle C_6 for the γ_r set D_{20} , choose the vertex $u_{11} \in C_6$, then the vertices u_{12}, u_{21} , u_{61} are dominated, since the vertex u_{12} is adjacent only to the vertex u_{11} and u_{13} , u_{12} becomes an isolate hence also choose the vertex u_{12} for the γ_r set D_{20} , since the vertices u_{23} and u_{63} are chosen for the γ_r set D_{20} the vertices u_{21}, u_{22} and u_{61}, u_{62} are adjacent in V – D_{20} . Proceeding similarly, choose the set of all vertices attached to the vertex v_4, thus by choosing the set of vertices $u_{41}, u_{42}, u_{43}, u_{33}, u_{53}, u_{63}$ for the γ_r set D_{20} the vertices u_{31}, u_{32} and $u_{51}, u_{52}, u_{61}, u_{62}$ are adjacent in V – D_{20} . Thus the set $D_{20} = \{u_{11}, u_{12}, u_{13}, u_{23}, u_{33}, u_{41}, u_{42}, u_{43}, u_{53}, u_{63}\}$ and the set $V - D_{20} = \{ u_{21}, u_{22}, u_{31}, u_{32}, u_{51}, u_{52}, u_{61}, u_{62}\}$ forms the γ_r set of G_{22} with cardinality $k_{20} = \gamma_r$. Hence G_{22} is k_{20} - γ_r enresdowed. Consider the set D_{21} of cardinality $k_{21} = \gamma_r + 1$, then these exists following subcases

Subcase (ii)(b_1) Consider the set $D_{21,1} = D_{20} \cup \{u_{pr_i}\}$, p = 2,3,5,6, r_i = 2, then the vertex $\{u_{pr_i}\}$, belong to the path P_t of G_{22} , without loss of generality, assume that $u_{pr_i} = u_{22}$ for $p = r_i = 2$, then the set V – $D_{21,1}$ is{ $u_{21}, u_{31}, u_{32}, u_{51}, u_{52}, u_{61}, u_{62}$ }, since u_{21} is adjacent with u_{31} and u_{51} is adjacent with u_{61}, thus all the other vertices in V – $D_{21,1}$ is also adjacent. Thus the set $D_{21,1}$ forms the restrained dominating set containing the minimum restrained dominating set of cardinality $k_{21,1} = \gamma_r + 1$. Hence G_{22} is $k_{21,1}$ - γ_r enresdowed.

Subcase (ii)(b_2) Consider the set $D_{21,2} = D_{20} \cup \{u_{pr_i}\}$, p = 2,3,5,6, r_i = 1, then the vertex u_{pr_i} belongs to the cycle C_n of G_{22} . Without loss of generality, assume that u_{pr_i}=u_{31}, then the set V – $D_{21,2}$ is {$u_{21}, u_{22}, u_{32}, u_{51}, u_{52}, u_{61}, u_{62}$ }, then the vertex u_{32} is an isolate in V – $D_{21,2}$. Thus the set $D_{21,2}$ is not a restrained dominating set and G_{22} is not $k_{21,2}$ - γ_r enresdowed. Consider a set D_{22} of cardinality $k_{22} = \gamma_r + 2$ where there exists the following subcases

Subcase (ii)(b_3) Consider the set $D_{22,1} = D_{20} \cup \{u_{p_1 r_{i1}} , u_{p_2 r_{i2}}\}$, P_1 , P_2 = 2,3,5,6. Such that $P_1 = P_2$, r_{i1}=1, $r_{i2} \neq 1$. Without loss of generality, assume that $u_{p_1 r_{i1}} = u_{21}$, $u_{p_2 r_{i2}} = u_{22}$, then the set V – $D_{22,1}$ is { $u_{31}, u_{32}, u_{51}, u_{52}, u_{61}, u_{62}$} which contains the set of adjacent vertices. Thus the set $D_{22,1}$ forms the restrained dominating set of cardinality $k_{22,1} = \gamma_r + 2$ containing the γ_r set D_{20} of G_{22} . Hence G_{22} is $k_{22,1}$ - γ_r enresdowed.

Subcase (ii)(b_4) Consider the set $D_{22,2} = D_{20} \cup \{u_{p_1 r_{i1}} , u_{p_2 r_{i2}}\}$, where P_1 , P_2 = 2,3,5,6 , such that $P_1 \neq P_2$, $r_{i1} = r_{i2} = 1$. Assume that $u_{p_1 r_{i1}} = u_{21}$, and $u_{p_2 r_{i2}} = u_{31}$, where u_{21} and u_{31} are adjacent, then the set V – $D_{22,2}$ is { $u_{22}, u_{32}, u_{51}, u_{52}, u_{61}, u_{62}$} where the vertices u_{22} and u_{32} forms the isolates in V – $D_{22,2}$, thus the set $D_{22,2}$ is not a restrained dominating set . Hence G_{22} is not $k_{22,2}$ - γ_r enresdowed.

Subcase (ii)(b_5) Consider the set $D_{22,3} = D_{20} \cup \{u_{p_1 r_{i1}} , u_{p_2 r_{i2}}\}$, P_1 , P_2 = 2,3,5,6 where $P_1 \neq P_2$, and r_{i1}=1, $r_{i2} \neq 1$. Without loss of generality, assume that $u_{p_1 r_{i1}} = u_{21}$, and $u_{p_2 r_{i2}} = u_{52}$, where the vertices u_{21} and u_{52} are not adjacent, then the set V – $D_{22,3}$ is { $u_{22}, u_{31}, u_{32}, u_{51}, u_{61}, u_{62}$} then the vertices u_{22} and u_{51} are isolates in V – $D_{22,3}$. Hence $D_{22,3}$ does not form a restrained dominating set. Hence G_{22} is not $k_{22,3}$ - γ_r enresdowed. Proceeding similarly, consider the set D_{23} of cardinality $k_{23} = n - 1$, where $D_{23} = D_{20} \cup \{ u_{21}, u_{22}, u_{31}, u_{32}, u_{51}, u_{52}, u_{61}\}$ and V – D_{23} = {u_{62}}. Then D_{23} is not a restrained dominating set of G_{22} . Hence G_{22} is not a k_{23} - γ_r enresdowed. Finally consider the set D_{24} , where $D_{24} = D_{23} \cup \{u_{62}\}$ is of cardinality k_{24} = n which contains the γ_r set D_{20} . Thus D_{24} forms a restrained dominating set of cardinality n containing the minimum restrained dominating set D_{20} of G_{22} . Hence G_{22} is k_{24} - γ_r enresdowed.

Subcase (ii)(c) In general, consider the graph $G_{2n_2} = C_nP_{n_i}$, for $n \geq 3$, where i = 3, 6, 9,….. . Let the vertex set of the cycle C_n be $\{v_1, v_2,…,v_i,…., v_s\}$and vertex set of the path P_{n_i} for i = 3, 6, 9,… be $\{u_{11}, u_{12}, ….., u_{1r_1},….., u_{1m_1}, u_{21}, u_{22},….., u_{2r_2},…...., u_{2m_2}, u_{31}, u_{32},…...., u_{3r_3},…....., u_{3m_3},….., u_{i1}, u_{i2},….., u_{ir_i},….., u_{im_i},…., u_{s1}, u_{s2},….., u_{sr_s},….., u_{sm_s}\}$, $1 \leq i \leq s$, $1 \leq r_i \leq m_i$, where the vertex $u_{11} = v_1$, $u_{21} = v_2$,……, $u_{i1} = v_i$,……, $u_{s1} = v_s$. Choose the vertex $u_{11} = v_1$, from the cycle, then the vertices u_{12}, u_{21}, u_{s1} are dominated. Similarly choose the vertex u_{14} from P_{t_1}, then the vertices u_{13} and u_{15} are dominated, where the vertex u_{12} and u_{13} are adjacent. Choose the vertex u_{17} from the path P_{t_1}, then the vertices u_{16} and u_{18} are dominated and where the vertices u_{15} and u_{16} are adjacent. Proceeding similarly, choose the vertices u_{1m_1-2}, u_{1m_1-1}, and $u_{1m_1}$from the path P_{t_1} which is attached to the vertex v_1.

Consider the path attached to $u_{21} = v_2$, since the vertex $u_{21}$is already dominated by the vertex u_{11} in C_n, without loss of generality, choose the vertex u_{23}, such that the vertices u_{22} and u_{24} are dominated and where the vertices u_{21} and u_{22} are adjacent. Proceeding similarly, choose the vertex u_{2m_2} from the path P_{t_2} attached to the vertex v_2. Choose the vertex $u_{41} = v_4$ from the cycle, then the vertices u_{42}, u_{31}, u_{51} are dominated, similarly choose the set of vertices $\{u_{44}, u_{47},…., u_{4m_4-2}, u_{4m_4-1}, u_{4m_4}\}$ from the path P_{t_4} which is attached to the vertex v_4 for the γ_r set D_{25} . Then the vertices $\{u_{42}, u_{43}, u_{45}, u_{46},…., u_{4m_4-4}, u_{4m_4-3}\}$ are adjacent in $V - D_{25}$.

Consider the path attached to the vertex $u_{31} = v_3$. Since the vertex u_{31} is already dominated by u_{41}, choose the vertex u_{33}, such that u_{31}, u_{32} are adjacent in $V - D_{25}$. Proceeding similarly, choose the end vertex u_{3m_3}. Similarly consider the path attached to the vertex $u_{51} = v_5$, where the vertex u_{51} is already dominated by u_{41}. Choose the vertex u_{53}, such that u_{51} and u_{52} are adjacent in $V - D_{25}$. Proceeding similarly, choose the vertex u_{5m_5}. Thus the γ_r set D_{25} is obtained, where $D_{25} = \{u_{11}, u_{14} , u_{17}, …….., u_{1m_1-2} , u_{1m_1-1}, u_{1m_1}, u_{23}, u_{26}, u_{29},…., u_{2m_2}, u_{33}, u_{36} , u_{39},…., u_{3m_3}, u_{41}, u_{44} , u_{47},….., u_{4m_4-2} , u_{4m_4-1}, u_{4m_4}, u_{53}, u_{56} , u_{59},…., u_{5m_5} ,…., u_{s3} , u_{s6} , u_{s9} ,…., u_{sm_s}\}$ and $V - D_{20} = \{u_{12} , u_{13} , u_{15} , u_{16}, …., u_{1m_1-4} , u_{1m_1-3} , u_{21} , u_{22} , u_{24} , u_{25}, u_{27} , u_{28} ,…….., u_{2m_2-2} , u_{2m_2-1}, u_{31}, u_{32}, u_{34}, u_{35}, u_{37}, u_{38},…., u_{3m_3-2} , u_{3m_3-1}, u_{42}, u_{43}, u_{45}, u_{46}, … , u_{4m_4-4} , u_{4m_4-3}, u_{51}, u_{52} , u_{54} , u_{55}, u_{57} , u_{58} ,…….., u_{5m_5-1} ,……, u_{s1} , u_{s2} , u_{s4}, u_{s5}, u_{s7}, u_{s8},…….., u_{sm_s-2} , u_{sm_s-1}\}$. Then the set D_{25} forms a minimum restrained dominating set of G_{2n_2} with cardinality $k_{25} = \gamma_r$. Hence G_{2n_2} is k_{25} - γ_r enresdowed.

Consider the set D_{26} of cardinality $k_{26} = \gamma_r + 1$, where there exists the following subcases.

Subcase (ii)(c_1) Consider the set $D_{26,1} = D_{25} \cup \{u_{pr_i}\}$, $1 \leq p \leq s$ and if p = 1,4,7,…, then r_i = 2,3,5,6,…, $u_p m_{p-3}$ or if p = 2,3,5,6,…, then r_i = 4,5,7,8,…, $u_p m_{p-1}$, where the vertex $\{u_{pr_i}\}$, belongs to the path P_{t_i}, $1 \leq i \leq s$ of G_{2n_2} . Since the vertex u_{pr_i} is adjacent only to the vertex u_{pr_i+1} in $V - D_{25}$, then there exists an isolates in $V - D_{26,1}$. Therefore $D_{26,1}$ is not a restrained dominating set of G_{2n_2} . Hence G_{2n_2} is not $k_{26,1}$ - γ_r enresdowed for any $k_{26,1} = \gamma_r + 1$.

Subcase (ii)(c_2) Consider the set $D_{26,2} = D_{25} \cup \{u_{pr_i}\}$, where $p \neq q + 1$, q = 0,3,6,9,….. and r_i = 1 then the vertex $\{u_{pr_i}\}$, belongs to the cycle C_n of G_{2n_2} , then there exists a set of vertices for p = 2,3,5,6,…, and r_i = 2,which forms an isolate in $V - D_{26,2}$. Hence $D_{26,2}$ is not a restrained dominating set of G_{2n_2} . Hence G_{2n_2} is not $k_{26,2}$ - γ_r enresdowed for any $k_{26,2} = \gamma_r + 1$.

Subcase (ii)(c_3) Consider the set $D_{26,3} = D_{25} \cup \{u_{pr_i}\}$, where $p \neq q + 1$, q = 0, 3,6,9,….. and r_i = 2, then the vertex $\{u_{pr_i}\}$ belong to the path P_{t_i}, $1 \leq i \leq s$ of G_{2n_2} and the vertex u_{p2} is adjacent to the vertex u_{p1} on the cycle C_n. Choose the vertex u_{pr_2} for the set $D_{26,3}$, then the set

V – $D_{26,3}$ exists and it is of cardinality $k_{26,3} = \gamma_r + 1$, where the set V – $D_{26,3}$ is {u_{12}, u_{13}, u_{15}, u_{16}, …., u_{1m_1-4}, u_{1m_1-3}, u_{21}, u_{24}, u_{25}, u_{27}, u_{28},……, u_{2m_2-2}, u_{2m_2-1}, u_{31}, u_{32}, u_{34}, u_{35}, u_{37}, u_{38},…., u_{3m_3-2}, u_{3m_3-1}, ………, u_{s1}, u_{s4}, u_{s5}, u_{s7}, u_{s8}, ……., u_{sm_s-2}, u_{sm_s-1}}then the vertex u_{21} is adjacent to u_{31}, similarly the vertex u_{51} is adjacent to u_{61}, and so on. Then there exists no isolates in V – $D_{26,3}$. Hence $D_{26,3}$ is a restrained dominating set containing the minimum restrained dominating set D_{25} of cardinality $k_{26,3} = \gamma_r + 1$. Hence G_{2n_2} is $k_{26,3}$ - γ_r enresdowed.

Consider a set D_{27} of cardinality $k_{27} = \gamma_r + 2$, then there exists the following subcases

Subcase (ii)(c_4) Consider the set $D_{27,1} = D_{25} \cup \{u_{p_1r_{i1}}, u_{p_2r_{i2}}\}$, where $p_1 = p_2$ and r_{i1}, $r_{i2} \neq$ p, for any p = q + 1, $r_{i1} \neq r_{i2}$, q = 0, 3,6,9,….. and p_1, p_2 = 1,4,7, …. , m_p, and for any p ≠ q + 1, q = 0, 3,6,9,….. with r_{i1}, r_{i2} = 3,6,9, …. , m_p. Choose the vertices $u_{p_1r_{i1}}$, $u_{p_2r_{i2}}$ in such a way they are adjacent in V – D_{25}, then there exists no isolate vertex in V – $D_{27,1}$. Hence the set $D_{27,1}$ forms a restrained dominating set of cardinality $k_{27,1} = \gamma_r + 2$, containing the minimum restrained dominating set D_{25} of G_{2n_2}. Thus G_{2n_2} is $k_{27,1}$ - γ_r enresdowed.

Subcase (ii)(c_5) Consider the set $D_{27,2} = D_{25} \cup \{u_{p_1r_{i1}}, u_{p_2r_{i2}}\}$, $p_1 \neq p_2$ and where $r_{i1} = r_{i2}$ = 1, such that p_1, p_2 = p, where p ≠ q + 1, q = 0, 3,6,9,….. Thus the vertices $u_{p_1r_{i1}}$, $u_{p_2r_{i2}}$ belong to the cycle C_n. Choose the vertices $u_{p_1r_{i1}}$, $u_{p_2r_{i2}}$ in such a way they are adjacent in V – D_{25}, then there exists no isolate vertex in V – $D_{27,2}$. Thus the set $D_{27,2}$ forms a restrained dominating set of cardinality $k_{27,2} = \gamma_r + 2$ with a γ_r set D_{25} of G_{2n_2}. Hence G_{2n_2} is $k_{27,2}$ - γ_r enresdowed.

Subcase (ii)(c_6) Consider the set $D_{27,3} = D_{25} \cup \{u_{p_1r_{i1}}, u_{p_2r_{i2}}\}$, $p_1 = p_2$, where $r_{i1} = r_{i2} \neq 2$, such that p_1, p_2 = p, where p ≠ q + 1, q = 0, 3,6,9,….. and r_{i1} = 1, r_{i2}= 4,5,7,8,…. , then the vertex $u_{p_1r_{i1}}$ belongs to the cycle C_n and $u_{p_2r_{i2}}$ belongs to the path P_{t_i}, 1 ≤ i ≤ s of G_{2n_2}, then there exists isolate vertices in V – $D_{27,3}$, such that the set $D_{27,3}$ is not a restrained dominating set. Hence G_{2n_2} is not $k_{27,3}$ - γ_r enresdowed.

Proceeding similarly, consider the set D_{28} of cardinality k_{28} = n – 1, whereD_{28}=D_{25} ∪{u_{13}, u_{15}, u_{16},…., u_{1m_1-4}, u_{1m_1-3}, u_{21}, u_{22}, u_{24}, u_{25}, u_{27}, u_{28},….., u_{2m_2-2}, u_{2m_2-1}, u_{31}, u_{32}, u_{34}, u_{35}, u_{37}, u_{38},……., u_{3m_3-2}, u_{3m_3-1}, u_{42}, u_{43}, u_{45}, u_{46}, ………, u_{4m_4-4}, u_{4m_4-3}, u_{51}, u_{52}, u_{54}, u_{55}, u_{57}, u_{58},…….., u_{5m_5-1},……, u_{s1}, u_{s2}, u_{s4}, u_{s5}, u_{s7}, u_{s8},……., u_{sm_s-2}, u_{sm_s-1}} and V – D_{28} = {u_{12}}. Thus the set D_{28} is not a restrained dominating set of G_{2n_2}. Hence G_{2n_2} is not a k_{28} - γ_r enresdowed. Finally consider the set D_{29}, where the set $D_{29} = D_{28} \cup \{u_{12}\}$ is of cardinality k_{29} = n, which contains the γ_r set D_{25} .Thus D_{29} forms a restrained dominating set of cardinality n. Hence G_{2n_2} is k_{29} - γ_r enresdowed.

Case (iii) Consider the graph $G_{3n_3} = C_nP_{n_i}$, 1 ≤ i ≤ n, where C_n, n ≥ 3 be the cycle and P_{n_i}, 1 ≤ i ≤ n be a path of G_{3n_3}. Let $P_{n_i} = P_{3t_1+1}$, for $t_1 \geq 1$. The cycle C_n, n ≥ 3 is attached with a paths of the type P_4, P_7, P_{10}….., then there exists the following subcases

Subcase (iii)(a) Consider the graph $G_{3n_3} = C_nP_{n_i}$, 1 ≤ i ≤ n, with n ≥ 3 and $P_{n_i} = P_{3t_1+1}$, for $t_1 \geq 1$. The vertex set of the cycle C_n be {v_1, v_2,…,v_i,…., v_n}, 1 ≤ i ≤n and the path P_{n_i} be {u_{11}, u_{12},….., u_{1r_1},….., u_{1m_1}, u_{21}, u_{22},….., u_{2r_2},….., u_{2m_2}, u_{31}, u_{32},…..,u_{3r_3},….., u_{3m_3},……, u_{i1}, u_{i2},….., u_{ir_i},….., u_{im_i}, ……., u_{n1}, u_{n2},….., u_{nr_n},….., u_{nm_n}}, 1 ≤ i ≤n , 1 ≤ r_i ≤ m_i, where the vertex $u_{11} = v_1$, $u_{21} = v_2$,……, $u_{i1} = v_i$,……, $u_{n1} = v_n$. Choose the set of vertices u_{11}, u_{21}, u_{31}, …., u_{n1} for the γ_r set D_{30} of G_{3n_3}, then the set of vertices u_{12}, u_{22}, u_{32}, …., u_{n2} are dominated, similarly choose the set of all vertices u_{14}, u_{24}, u_{34},…., u_{n4} for the γ_r set D_{30} of

G_{3n_3}, then the set of vertices $u_{13}, u_{23}, u_{33}, \ldots, u_{n3}$ are dominated and thus the vertices $u_{12}, u_{22}, u_{32}, \ldots, u_{n2}$ and $u_{13}, u_{23}, u_{33}, \ldots, u_{n3}$ are adjacent in $V - D_{30}$. Proceeding similarly choose the set of vertices $u_{n1}, u_{n2}, \ldots, u_{nr_n}, \ldots, u_{nm_n}$, then the set $D_{30} = \{u_{i1}, u_{i4}, \ldots, u_{ir_i}, \ldots, u_{im_i}\} 1 \leq i \leq n, 1 \leq r_i \leq m_i$. Thus the set D_{30} forms the minimum restrained dominating set of G_{3n_3} with cardinality $k_{30} = \gamma_r$. Hence G_{3n_3} is k_{30} - γ_r enresdowed.

Consider any set D_{31} of cardinality γ_r+1, then the set $D_{31} = D_{30} \cup \{u_{pr_i}\}, 1 \leq p \leq n, r_i \neq q + 1, q = 0, 3, 6, \ldots$ Since each vertex u_{pr_i} is adjacent only to the vertex $u_{p(r_i+1)}$ in $V - D_{30}$, then there exists an isolate in $V - D_{31}$. Therefore D_{31} is not a restrained dominating set. Hence G_{3n_3} is not k_{31} - γ_r enresdowed for any $k_{31} = \gamma_r + 1$.

Consider any set D_{32} of cardinality $\gamma_r + 2$, then there exists the following subcases.

Subcase (iii)(a_1) Consider the set $D_{32,1} = D_{30} \cup \{u_{p_1 r_{i1}}, u_{p_2 r_{i2}}\}, 1 \leq p_1, p_2 \leq n, r_{i1}, r_{i2} \neq q + 1$ for $q = 0,3,6,\ldots$ such that $p_1 = p_2$ and $r_{i1} \neq r_{i2}$. The set $D_{32,1}$ is of cardinality $k_{32,1} = \gamma_r + 2$. Choose the vertices $u_{p_1 r_{i1}}, u_{p_2 r_{i2}}$ in such a way that they are adjacent in $V - D_{30}$ and thus there exists no isolates in $V - D_{32,1}$. Thus the set $D_{32,1}$ forms the restrained dominating set of cardinality $k_{32,1} = \gamma_r + 2$ which contains the γ_r set D_{30}. Hence G_{3n_3} is $k_{32,1}$ - γ_r enresdowed.

Subcase (iii)(a_2) Consider the set $D_{32,2} = D_{30} \cup \{u_{p_1 r_{i1}}, u_{p_2 r_{i2}}\}, 1 \leq p_1, p_2 \leq n, r_{i1}, r_{i2} \neq q + 1$ for $q = 0,3,6,\ldots$, such that $p_1 \neq p_2$. The set $D_{32,2}$ is of cardinality $k_{32,2} = \gamma_r + 2$. Since $p_1 \neq p_2$, the vertices $u_{p_1 r_{i1}}$ and $u_{p_2 r_{i2}}$ belongs to two distinct paths which is not adjacent in $V - D_{30}$. Since by considering any vertex $u_{p_1 r_{i1}}$ from $V - D_{30}$ for $D_{32,2}$ where it leaves the vertex $u_{p_1(r_{i1}+1)}$ as an isolate in $V - D_{32,2}$. Therefore the set $D_{32,2}$ is not a restrained dominating set of G_{3n_3}. Hence G_{3n_3} is not $k_{32,2}$ - γ_r enresdowed.

Proceeding similarly, consider the set D_{33} of cardinality $k_{33} = n - 1$, which is not a restrained dominating set of G_{3n_3}. Hence G_{3n_3} is not k_{33} - γ_r enresdowed. Finally consider the set D_{34}, where $D_{34} = D_{33} \cup \{u_{p_1 r_{i1}}\}$, such that $u_{p_1 r_{i1}}$ does not belong to D_{33}, then the set D_{34} form a restrained dominating set of cardinality $k_{34} = n$, where it contains the minimum restrained dominating set D_{30} of G_{3n_3}. Thus G_{3n_3} is k_{34} - γ_r enresdowed. Hence G_{3n_3} is k - γ_r enresdowed for any k, where $\gamma_r + l \leq k \leq n$, for even $l \geq 0$, except for $k = n - 1$.

Subcase (iii)(b) Consider the graph $G_{3n_3} = C_n P_{n_i}, 1 \leq i \leq n$, with $n \geq 3$ and $P_{n_i} = P_{3t_1+1}$, for $t_1 \geq 1$. Let D_{35} be the γ_r set of G_{3n_3}. Choose the vertex u_{11}, for the γ_r set D_{35}, then in any cycle, the vertex adjacent to u_{11} is u_{12}, u_{21}, u_{n1} are dominated, similarly choose the vertex u_{14}, such that the vertices u_{12} and u_{13} are dominated and they are adjacent in $V - D_{35}$, also choose the vertex u_{17}, such that the vertices u_{15} and u_{16} are adjacent in $V - D_{35}$. Proceeding similarly, choose u_{1m_1} for the γ_r set D_{35}. Consider another path which contains the vertex u_{21} which is dominated by u_{11}, since u_{21} is dominated, choose u_{23} and u_{26} for the γ_r set D_{35} such that the vertices u_{22}, u_{24} and u_{25} are dominated by the vertices u_{23} and u_{26}. Similarly choose the vertices u_{1m_1-1}, u_{1m_1} for the γ_r set D_{35}. Proceeding similarly, the set D_{35} forms a minimum restrained dominating set of cardinality $k_{35} = \gamma_r$. Hence G_{3n_3} is k_{35} - γ_r enresdowed.

Consider any set D_{36} of cardinality $k_{36} = \gamma_r+1$, since the vertices $u_{11}, u_{41}, \ldots$ are chosen for the γ_r set D_{35}, where the vertices u_{21}, u_{31} are adjacent in $V - D_{35}$. Thus there exists an vertices u_{21}, u_{22}, u_{31} which are adjacent in $V - D_{35}$. Then there exists following types of sets.

Subcase (iii)(b_1) Consider the set $D_{36,1}$ of cardinality $k_{36,1} = \gamma_r+1$, where $D_{36,1} = D_{35} \cup \{u_{p_1 r_{i1}}\}$, where $u_{p_1 r_{i1}} = v_i$, v_i belongs to the cycle C_n and does not belong to the set D_{35}. Consider any vertices u_{21}, u_{22}, u_{31}, where if the vertex $u_{21} = v_2$ is considered for the set $D_{36,2}$, where suppose if $u_{p_1 r_{i1}} = u_{21}$, then the vertices u_{22} and u_{31} are isolates in $V - D_{36,2}$. Thus the set $D_{36,1}$ does not form a restrained dominating set. Hence G_{3n_3} is not $k_{36,1}$ - γ_r enresdowed. Subcase (iii)(b_2) Consider the set $D_{36,2} = D_{35} \cup \{u_{p_2 r_{i2}}\}$, where $u_{p_2 r_{i2}}$ belong to the path P_{n_i} of G_{3n_3}. Since every vertex $u_{p_2 r_{i2}}$ in P_{n_i} is adjacent only to $u_{p_2(r_{i2}+1)}$ .By choosing the vertex $u_{p_2 r_{i2}}$ there exists an isolate vertex $u_{p_2(r_{i2}+1)}$ in V – $D_{36,2}$. Thus the set $D_{36,2}$ is not a restrained dominating set. Hence G_{3n_3} is not $k_{36,2}$ - γ_r enresdowed for any cardinality $k_{36,2}$ = γ_r+ 1 of G_{3n_3}.

Subcase (iii)(b_3)

Consider the set $D_{36,3}$ of cardinality $k_{36,3}$= γ_r+1, where the set $D_{36,3} = D_{30} \cup \{u_{p_3 r_{i3}}\}$, where the vertex $u_{p_3 r_{i3}}$ is adjacent to any vertex v_i, such that $u_{p_3 r_{i3}} = u_{22}$. Since the vertices u_{21}, u_{22}, u_{31} are adjacent in G_{3n_3}. By choosing u_{22} for the set D_{33} , there exists no isolates in V – $D_{36,3}$. Thus the set $D_{36,3}$ forms the restrained dominating set of G_{3n_3} containing the minimum restrained dominating set D_{35}. Hence G_{3n_3} is $k_{36,3}$ - γ_r enresdowed. Proceeding similarly, consider the set D_{37} of cardinality k_{37} = n – 1, which is not a restrained dominating set of G_{3n_3}. Hence G_{3n_3} is not k_{37} - γ_r enresdowed. Finally consider the set D_{38} of cardinality k_{38} = n and G_{3n_3} is k_{38} - γ_r enresdowed. Hence G_{3n_3} is k - γ_r enresdowed for any k, where $\gamma_r \le k \le$ n, except for k = n – 1.

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