International Journal of Technical Innovation in Modern Engineering & Science (IJTIMES)
Impact Factor: 5.22 (SJIF-2017), e-ISSN: 2455-2585
Volume 5, Issue 07, July-2019

SITNIKOV FIVE BODY PROBLEM FORMING SQUARE CONFIGURATION UNDER PERTURBATION OF PHOTOGRAVITATION: AN ELLIPTIC CASE

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Abstract— This paper deals with averaging the equation of motion of the elliptic Sitnikov restricted five-body problem when all the primaries as source of same radiation pressure. We assumed that the primaries are at the vertices of a square so the distances of the primaries from centre of mass are time depending. Next we have developed averaged equation of motion by applying the Van der Pol transformation and averaging technique of Guckenheimer and Holmes (Nonlinear Oscillations, Dynamical System Bifurcations of Vector Fields, Springer, Berlin (1983). In addition to the resonance criterion at the 3/2 commensurability we have chosen, is the angular velocity of the coordinate system. Furthermore we have linearized the equation of motion to obtain the Hill's type equation, then by using the Floquet's theory (Jose and Saletan 1998) to find the approximate solution. Finally the solutions are obtained by the Poincare surface of sections. It is shown that chaotic region emerging from the destroyed invariant tori, can easily be seen for certain eccentricities.

Keywords— Sitnikov Problem, Square Configuration, Poincare Method, Average of Equation of Motion, Van der Pol transformation.

I. Introduction

In the present paper, we have studied the Sitnikov five–body problem when all the primaries are moving in elliptic orbits around their common center of mass. The Sitnikov problem is a particular case of the restricted three-body problem in which two primaries with equal masses $\left(m_1 = m_2 = m = \frac{1}{2}\right)$ are moving in a circular orbit or an elliptic orbits around their common center of mass under the Newtonian force of attraction and the infinitesimal mass m_3 (the infinitesimal mass is much less than the masses of the other two primaries) is moving along the line perpendicular to the plane of motion of the primaries and passing through the center of mass of the primaries. It was Pavanini (1907) who originally introduced this problem by taking the circular case. The circular problem was discussed in detail by MacMillan (1913) where he showed that the exact solution can be expressed in terms of Jacobi elliptic functions.

Sitnikov (1960) studied the existence of oscillating motion of the three-body problem (a detailed description of this work can be found in Stumpff, 1965). Sitnikov's problem has further been studied by many authors. Perdios and Markellos (1988) have studied stability and bifurcations of Sitnikov motions. Liu and Sun (1990) have studied the Sitnikov problem without taking the original differential equation and discovered an invariant set of hyperbolic solutions whereas Hagel (1992) has studied the problem by a new analytic approach. Belbruno *et al* (1994) have studied the family of periodic orbits which bifurcate from the circular Sitnikov problem. Jalali and Pourtakdoust (1997) have studied the regular and chaotic solutions of the Sitnikov problem near the 3/2 commensurability.

Dvorak and Sun (1997) have studied the phase space structure of the extended Sitnikov problem. Chasley (1999) has studied the global analysis of the generalized Sitnikov problem. Corbera and Llibre (2000) have proved the existence of symmetric periodic orbits of the Sitnikov problem. Lara and Buendia (2001) have studied symmetries and bifurcations in the Sitnikov problem. Faruque (2002) has studied the axial oscillation of a planetoid in the restricted three-body problem. Faruque (2003) has established the new analytical expression for the position of the infinitesimal body in the elliptic Sitnikov problem. Perdios (2007) has studied the manifolds of families of three-dimensional periodic orbits in the Sitnikov three-body problem. Soulis *et al* (2007) have studied stability of motion in the Sitnikov three-body problem.

Perdios *et al* (2007) has studied the straight line oscillations of the Sitnikov family of the photo-gravitational circular restricted three-body as well as the associated families of 3-D periodic orbits. Soulis *et al* (2008) have investigated periodic orbits and bifurcation in the Sitnikov four-body problem with primaries as point masses. Perdios and Markellos (2012) have studied the self resonant bifurcations of the Sitnikov family and the appearance of 3-D isolas in the restricted three-body problem. Douskos *et al* (2012) have studied on Sitnikov-like motions generating new kinds of 3-D periodic orbits in the restricted three-body problem with prolate primaries.

In the present work, we proposed to discuss the five-body problem in the sense of Sitnikov. Here we consider that the primaries are moving in elliptic orbits around their common center of mass with the assumption that all the primaries be at the vertices of a square for all time. Also averaging of the equations of motion using Van der Pol transformation is under proposal. Furthermore the equation of motion is transformed to the Hill's equation and has determined the solution. Finally the Poincare surface of section has been compared by increasing the eccentricity of the primaries. For chaos exploration, the method of sections has been used to predict stochastic behavior near hyperbolic points. In the proposed work, invariant manifolds are of heteroclinic type that connects hyperbolic fixed points.

II. Equation of Motion

Let P_1, P_2, P_3 and P_4 be the four radiating primaries of equal masses $m_1 = m_2 = m_3 = m_4 = m = \frac{1}{4}$, all the four radiating primaries are the vertices of a square and moving in elliptic orbits around their center of mass *O* which is taken as origin and $OP_1 = OP_2 = OP_3 = OP_4 = r(t)$ as the distances of primaries from their center of mass. The infinitesimal mass m , which is much less than the masses of the primaries. The infinitesimal is confined to a motion perpendicular to the plane of motion of the four primaries which are equally far away from the barycenter of the system. The infinitesimal is moving along a straight line perpendicular to the plane of motion of the primaries and passing through the center of mass of the primaries. In such a system the motion of the infinitesimal is one dimensional.

Since the primaries are at the vertices of a square and moving in elliptic orbits around their common center of mass of the system hence the distances between the primaries, vary with the time but always in such a way that their mutual distances apart remain in the same ratio. Obviously the distances of primaries from the center of mass of the system at any time remain in the same ration. In such a way the lines joining the primaries always form a square configuration with variable sides.

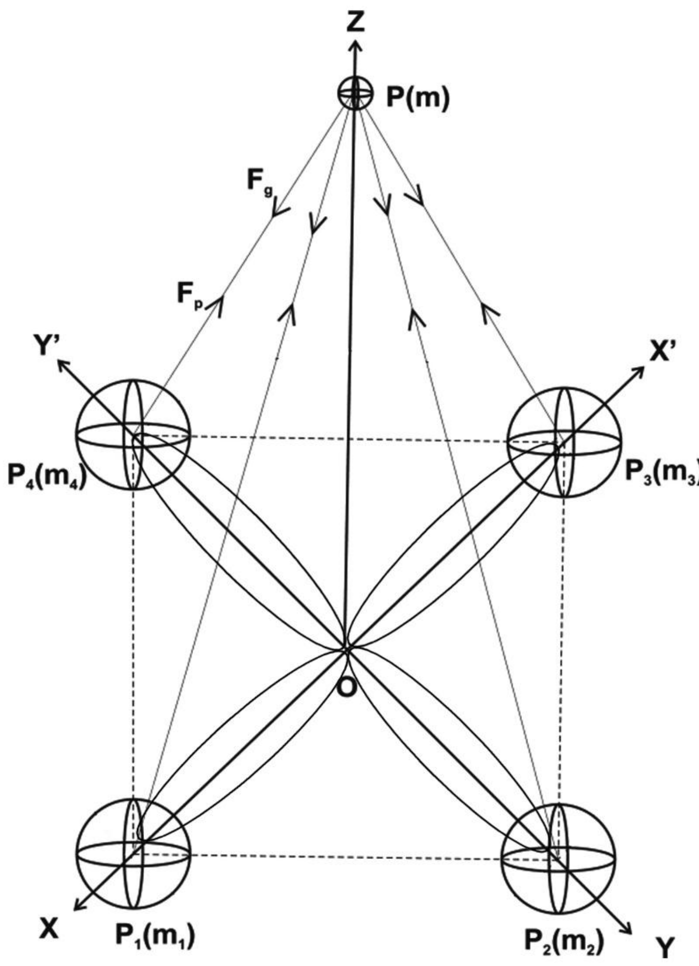

Figure : Sitnikov Five body Problem

Let us consider the common center of mass of the primaries as the common focus of the elliptic orbits of the primaries and the line OP_1 and OP_2 as the *x-axis* and *y-axis* respectively and the line of motion of the infinitesimal mass as the *z-axis*. Clearly the coordinates of the primaries are given by

$$x_1 = r(t), x_2 = x_4 = 0, x_3 = -r(t),$$
$$y_1 = y_3 = 0, y_2 = r(t), y_4 = -r(t),$$
$$z_1 = z_2 = z_3 = z_4 = 0,$$

where origin *O* is treated as common focus of all the elliptic orbits.

Let at any time *t*, *f be* the true anomaly of the first primary moving in elliptic orbit then the angular velocity of the plane of motion of the primaries around *z-axis* is $\dot{f}$ .The equation of motion the infinitesimal mass *P(x, y, z)* under gravitational field of the four radiating primaries can be formulated in Cartesian barycentric co-ordinate system *(O, XYZ).*

If $\vec{F}_1 = (i = 1, 2, 3, 4)$ be the gravitational forces extended on *m* due to m_i along $\overrightarrow{PP_i}, \vec{F}_{p_i}$ be the radiation pressure on m_5 due to m_i along $\overrightarrow{P_i P}$ and $F_i\left(1 - P_i^{'}\right)$ the total forces extended on m_5 due to m_i along $\overrightarrow{PP_i}$, , then the equations of motion of the infinitesimal mass are

$$\ddot{x} - 2\dot{f}\dot{y} - \dot{f}^2 - \ddot{f}y = -\sum_{i=1}^{4} \frac{G\left(1 - P_i{'}\right)\left(x - x_i\right)m_i}{r_i^3},$$
$$\ddot{y} + 2\dot{f}\dot{x} - \dot{f}^2 y + \ddot{f}y = -\sum_{i=1}^{4} \frac{G\left(1 - P_i{'}\right)\left(y - y_i\right)m_i}{r_i^3}, \quad (1)$$
$$\ddot{z} = -\sum_{i=1}^{4} \frac{G\left(1 - P_i{'}\right) z m_i}{r_i^3},$$

where

$$P_i{'} = \frac{\text{Radiating Pressure due to the Primary } P_i}{\text{Gravitational Force due to the Primary } P_i{'}},$$
$$0 < P_1{'}, P_2{'} P_3{'} P_4{'} << 1,$$
$$r_i^2 = \left(x - x_i\right)^2 + \left(y - y_i\right)^2 + z^2.$$

Since, the motion of the infinitesimal m_5 in the Sitnikov problem is only along *z*-axis hence by fixing the unit of time by $G = 1$, total mass of the primaries as mass unit, the equation of motion is obtained from the Equation (1) by putting $x(t) = y(t) = 0,$ as

$$\frac{d^2 z}{dt^2} = \frac{z\left(1 - P^*\right)}{\left\{z^2 + r(t)^2\right\}^{\frac{3}{2}}} \quad (2)$$

where $P^* = \frac{1}{4}\sum_{i=1}^{4} P_i{'}.$

Since all the primaries are sources of same radiation pressure, we assume that $P_1{'} = P_2{'} = P_3{'} = P_4{'}$. Here z is the dimensionless distance of the infinitesimal mass from the barycenter of the systems and t is representing time, so $r(t)$ is given by the solution of the transcendental Kepler's equation and given as an infinite power series in the primaries eccentricity e (Brouwer and Clemence 1961)

$$r(t) = a\left\{1 + \sum_{n=1}^{\infty} r_n(t) e^n\right\} \quad (3)$$

where $r_1(t) = -\cos t, r_2(t) = \frac{1}{2}(1 - \cos 2t), r_3(t) = \frac{3}{8}(\cos t - \cos 3t), \ldots$

Here *a* is the semi-major axis of the elliptic orbit, which we have taken $\frac{1}{\sqrt{2}}$. This expansion is given by Stumpff (1965) and is convergent for e < 1. It is also used by Hagel (1992).

Now the solution of (Equation 2) is plotted over the different time interval and the radiation parameter. Here we plot for the value, e = 0.3, and P. To observe the nature of the graph, we have plotted the same graph for the same value but for long time of interval 0 < t < 100.

If $e \neq 0$ and $z_{\max} \leq r(t)$ for all t , we expand the right hand side of Equation (2) about *z* to obtain infinite power series solution and truncating them after certain order of *z*. Taking the first nonlinear term as z^3.

$$\frac{d^2z}{dt^2} + (1-P^*)\left\{\frac{z}{r(t)^3} - \frac{3z^3}{2(t)^5}\right\} = 0 \tag{4}$$

Keeping the terms proportional to the $e^{\alpha'}z^{\alpha'}$ where $\alpha' + \alpha'' \leq 5$ (α' and α'' are positive integers). Thus the Equation (4) may be written as

$$\begin{aligned} \frac{d^2z}{dt^2} &= -\varpi^2 z - kP^* f(z,t) + P^*\left(f(z,t) + \varpi^2 z\right), \\ &= -\varpi^2 z - \varepsilon f^*(z,t). \end{aligned} \tag{5}$$

Writing the equation of motion in phase-space by the transformation

$$\begin{aligned} \dot{z} &= s, \\ \dot{s} &= -\varpi^2 z - \varepsilon f^*(z,t). \end{aligned} \tag{6}$$

where

$$\begin{aligned} f^*(z,t) &= \left(1 - \frac{1}{k}\right) f(z,t) - \frac{1}{k}\varpi^2 z, \\ f(z,t) &= f_1(t)z + f_2(t)z^3, \\ f_1(t) &= 6\sqrt{2e}\cos t + 9\sqrt{2e^2}\cos 2t + \frac{\sqrt{2}e^3}{4}(27\cos t + 53\cos 3t) \\ &\quad + 7\sqrt{2}e^4(4\cos 2t + 11\cos 4t), \\ f_2(t) &= -6\sqrt{2} - 30\sqrt{2}e\cos t - 30\sqrt{2}e^2(1 + 2\cos 2t), \\ \varpi^2 &= 2\sqrt{2} + 3\sqrt{2}e^2 + \frac{15\sqrt{2}e^4}{4}, \\ kP^* &= \varepsilon. \end{aligned} \tag{7}$$

Multiple ε is introduced as a perturbation parameter which is eventually will be set to be unity (Jalali and Pourtakdoust 1997).

III. Averaging of the Equation of Motion

The solution of the Equation (6) may be obtained by successive application of the Van der Pol transformation and the averaging technique of Guckenheimer and Holmes (1983). The equation of motion can be extended to (z, s) space and is transformed into a rotating co-ordinates system with angular velocity $\omega = \frac{2\alpha\pi}{T}$. Let us introduce an invertible transformation by

$$\begin{pmatrix} z \\ s \end{pmatrix} = \begin{pmatrix} \cos \varpi t & -\sin \varpi t \\ -\omega \sin \varpi t & -\omega \cos \varpi t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Using Equations (6) and (7), solving for $\dot{x}_1, \dot{x}_2$, we get

$$M\left(x_1, x_2, t, \varepsilon\right) = \left\{ \left(\frac{1}{\varepsilon}\left(\omega^2 - \varpi^2\right) z - f^*\left(z, t\right) z^3 \right) \right\} \tag{8}$$

Now with the help of above equations, we get the averaged equations

$$\bar{M}_1\left(y_1, y_2\right) = \frac{1}{T}\int_0^T -\frac{1}{\omega}\left[\left\{\left(\omega^2 - \varpi^2\right) z - f^*\left(z, t\right)\right\}\right] \sin \varpi t dt$$

and $$\bar{M}_2\left(y_1, y_2\right) = \frac{1}{T}\int_0^T -\frac{1}{\omega}\left[\left\{\left(\omega^2 - \varpi^2\right) z - f^*\left(z, t\right)\right\}\right] \cos \varpi t dt. \tag{9}$$

Thus the required equation of motion corresponding to the averaged Hamiltonian can be written as

$$y_1' = \frac{\partial H}{\partial y_2}, \quad y_2' = \frac{\partial H}{\partial y_1}. \tag{10}$$

IV. Discussions and Conclusion

In the square configuration the Poincare surface of sections of the Sitnikov five-body problem have been discussed graphically and then they have been compared by Jalali and Pourtakdoust (1997). It was shown that chaotic sea emerging from the destroyed invariant manifolds can easily be seen for certain eccentricities. By increasing the eccentricity of the primaries to 0.21, chaotic motion occurs (not clearly visible) in the vicinity of the hyperbolic points. From the above discussions we conclude that periodic tube appear in the eccentricity $e = 0.18$. By increasing the eccentricity of the primaries, periodic tube disappears and the chaotic regions occur.

In the case of Jalali and Pourtakdoust (1997), the two primaries are moving in elliptic orbits around their common center of mass. From the discussion of Poincare surface of section, the results of the numerical integration not only reflect many qualitative characteristics of the system, but also confirm the obtained analytical solutions in the eccentricity $e = 0.03$. The chaotic motion occurs in the vicinity of the hyperbolic points by increasing the eccentricity to $e = 0.055$ and by further increasing the eccentricity to $e = 0.08$, surrounding periodic tubes disappear and the main stochastic region joins the escaping one with a sampling time equal to 3π .

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