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Solution of Lane Emden Equation for Polytropic Index n= 1 using Numerical Iterative Series Approximation Method

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Abstract— Lane Emden Equation describes the Newtonian equilibrium structure of a self-gravitating polytropic fluid sphere. It is used to model stellar interiors and stars clusters etc. In this paper we have used newly developed Numerical Iterative Series Approximation Method to solve Linear LEE of polytropic index one. The results obtained in this work are also compared with numerical results obtained using formula which is used as a yardstick for testing NISAM. Good agreement is observed between the present results and the numerical results.

Keywords— Polytropes , Numerical iterative Series Approximation Method

I. Introduction

Polytropic stellar models are the most basic type of stellar models where stellar structure equation is established by assuming the star to be spherically symmetric and that does not have a magnetic field .(Chandershekhar,1939)

For a spherical symmetric star, the gravitational force acting on the plasma causes a pressure-stratification profile, since the pressure at a given point is due to the weight of the gas above it, pressure increases as a function of depth. At equilibrium, the pressure gradient present in stars counterbalances the gravitational force which leads to hydrostatic balance governed by the equation

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2} \quad [1]$$

$dP(r)$ is the variation of pressure with radius, $M(r)$ is mass within a sphere of radius r, $\rho(r)$ density. In polytropic stellar model, Pressure P and density (ρ) is related by the relation $P = K\rho^{(n+1)/n}$ and n is polytropic index. It was first considered by Lane (1870) but the same problem was independently considered by Ritter (1878) and also by Kelvin.

Plugging the value of $M(r)$ as $4\pi r^2 \rho dr$ from equation of conservation of mass and differentiating equation [1] with respect to r gives

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP(r)}{dr}\right) = -4\pi G\rho(r) \quad [2]$$

By introducing a dimensionless $y(r)$ where $[y(r)]^n = \frac{\rho(r)}{\rho_c}$ where ρ_c stands for the density at the center of the star. This transforms eq. [2] to

$$(n+1)\frac{K\rho_c^{\frac{1-n}{n}}}{4\pi G}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dy(r)}{dr}\right) + [y(r)]^n = 0 \quad [3]$$

By introducing another dimensionless variable ' x ' defined by the relation

$$r=\left[(n+1)\frac{K\rho_c^{\frac{1-n}{n}}}{4\pi G}\right]^{\frac{1}{2}}x$$

which finally yields the following

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{dy}{dx}\right)=-y^n \qquad [4]$$

This is desired **Lane-Emden equation.** Being a second order differential equation requires two boundary conditions

$$y=1,\frac{dy}{dx}=0 \text{ When x= 0 (at the centre)} \qquad [5]$$

$$y=0,\frac{dy}{dx}=\text{Finite x=1(at the surface)} \qquad [6]$$

Unfortunately, the Lane-Emden equation does not have analytic solution for arbitrary values of n .In fact there are only three analytical solutions.

II. The Numerical Iterative Series Approximation Method

An exact solution only occurs for a few values of *n*, in other cases a numerical solution is required. We have used new iterative method to solve linear differential equation which was already solved with Pade` approximation [Pascual, 1977], Power series [Mohan & Al-Bayaty, 1980], Ritz method [He, 2003],The Variational iteration method [Dehghan & Shakeri,2008], Adomian decomposition Method [Rach,2014] and New Interative method [Rehman,2016] The **Numerical Iterative Series Approximation Method**, hereinafter will be referred NISAM. In proposed NISAM an ordinary non linear differential equation is converted into two simultaneous equations by suitable transformation

III. Solution of LEE using NISAM

Firstly converting non linear Lee equation to linear Lee equation

$$\text{Let } \frac{dy}{dx}=z\rightarrow y=\int_0^x z\,dx \qquad [7]$$

$$\frac{dz}{dx}=-\frac{2z}{x}-y\rightarrow z=\int_0^x\left(-\frac{2z}{x}-y\right)dx \qquad [8]$$

Using boundary condition $x_0=0,y_0=1,y_0^{`}=0$

$$y_1=\int_0^x z_0\,dx=\int_0^x 0.dx=C$$

But within the boundary condition y_1 has to be zero so C has to be zero only.

$$z_1=\int_0^x\left(-\frac{2z}{x}-y\right)dx=\int_0^x-\frac{2\times 0}{x}-(1)\,dx=0-x=-x$$

$$y_2 = \int_0^x -x\,dx = -\frac{x^2}{2}$$

$$z_2 = \int_0^x \left(\frac{2x}{x} - (0)\right) dx = 2x$$

$$y_3 = \int_0^x 2x\,dx = x^2$$

$$z_3 = \int_0^x -\frac{4x^2}{x} - \left(\frac{-x^2}{2}\right) dx = -4x + \frac{x^3}{2.3}$$

$$y_4 = \int_0^x (-4x + \frac{x^3}{6})dx = \frac{-4x^2}{2} + \frac{x^4}{2.3.4}$$

$$z_4 = \int_0^x -2\frac{(-4x + \frac{x^3}{6})}{x} - x^2 dx = 8x - \frac{2x^3}{2.3.3} - \frac{2x^3}{2.3}$$

$$y_5 = \int_0^x (8x - \frac{2x^3}{18} - \frac{2x^3}{6})dx = \frac{8x^2}{2} - \frac{2x^4}{2.3^2.4} - \frac{2x^4}{2.3.4}$$

$$z_5 = \int_0^x \frac{-2}{x}(\frac{-8x^2}{2} - \frac{2x^4}{72} - \frac{2x^4}{24}) - (\frac{-4x^2}{2} + \frac{x^4}{24})dx = -16x + \frac{4x^3}{2.3^3} + \frac{2x^3}{3^2} + \frac{2x^3}{3} - \frac{x^5}{2.3.4.5}$$

And so on

$$y = y_0 + y_1 + y_2 + y_3 + y_4 + y_{5.}\ldots\ldots\ldots\ldots\ldots\ldots.$$

Grouping and adding the equal power terms

$$y = 1 - \underbrace{\frac{x^2}{2} + x^2 - \frac{4x^2}{2} + \frac{8x^2}{2} - \frac{16x^2}{2}}_{\text{1st G.P Series}} + \underbrace{\frac{x^4}{2.3.4} - \frac{2x^4}{2.3^2.4} - \frac{2x^4}{2.3.4} + \frac{4x^4}{2.3^3.4} + \frac{2x^4}{3^2.4} + \frac{2x^4}{3.4}}_{\text{2nd G.P.series}} - \underbrace{\frac{x^6}{2.3.4.5.6} + \frac{2x^6}{2.3.4.5^2.6}}_{\text{3rd G.P series}} + ---$$

Summing the individual Geometrical Progression series results in the following series.

$$y = 1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040} + \ldots\ldots\ldots \qquad [9]$$

IV. Result and Discussion

Series obtained (equation 9) have been used to calculate numerical values of y(r) at different points. The result obtained is presented in Table 4.1 which represents the physical parameter y(r) calculated using NISAM whose values decreases as the scaled radius x increases and will become zero at the boundary which corresponds to actual condition of the star. The first column represent scaled radius of the stellar configuration, while second column represent the values obtained using **NISAM**.The numerical value obtained by NISAM at x=0.1,0.2,0.3,0.4,0.5and 0.6 exactly matches with result obtained by exact solution **.**So NISAM can be used to solve problems more efficiently.

Table:-4.1 Comparison between Authors value using NISAM and Exact value

For n=1			
x	**NISAM (y)**	**EXACT* (y)**	**ERROR**
0.1	0.998334	0.998334	0.0000000
0.2	0.993347	0.993347	0.0000000
0.3	0.985067	0.985067	0.0000000
0.4	0.973546	0.973546	0.0000000
0.5	0.958851	0.958851	0.0000000
0.6	0.941071	0.941071	0.0000000
0.7	0.920311	0.920311	0.0000002
0.8	0.896695	0.896695	0.0000005
0.9	0.870362	0.870363	0.0000012
1.0	0.841468	0.841471	0.0000027

Exact* value is calculated using the formula

Table 4.1 represents the physical parameter y(r) calculated using NISAM whose values decreases as the scaled radius x increases and will become zero at the boundary which corresponds to actual condition of the star.

4. Conclusion

In this paper, we have presented the NISAM for solving linear form of Lane-Emden equation. This is hybrid method comprising of new iterative method and series approximation method developed by us to solve Lane-Emden equation. This method reduces Lane-Emden Equation into a set of algebraic equations and the solution obtained has been compared with the exact solution. The value calculated using NISAM almost matches the exact values the percentage error ranging from 0.0 to 3.57×10^{-6} which is however very small. The results shows that method can be used to solve other problem salso efficiently

References

1. S. Chandrasekhar, "An Introduction to the Study of Stellar Structure", *Dover Publications*, New York, NY, USA, 1939.
2. Lane, Jonathan Homer ."On the Theoretical Temperature of the Sun under the Hypothesis of a Gaseous Mass Maintaining its Volume by its Internal Heat and Depending on the Laws of Gases Known to Terrestrial Experiment". *The American Journal of Science and Arts*. 2. 50: 57–74,1870.
3. Ritter, A. (1878). *Wiedemann Annalen*. 6, 135,1878.
4. P.Pascual,*Astron.Astrophys*.60,161-163,1977.
5. C. Mohan,A.R.Al-Bayaty,*Astrophys.SpaceSci*.73,227-239,1980
6. J.H. He, *Appl. Math. Comput*. 143,539-541,2003.
7. M. Dehghan, F.Shakeri. *New.Astron*.13,53-59,2008.
8. M.S.Rehman, M.J. Amir "New iterative method of solution of system of linear differential equations", *International journal of Science and Research*, vol. 5, Feb ,2016.