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# ON CERTAIN RESULTS INVOLVING RATIO OF INFINITE PRODUCTS AND MULTI SUMMATION FORMULAE

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***Abstract: - Some Rogers-Ramanujan multi sum identities can be expressed in terms of infinite products. In this paper, an attempt has been made to establish the certain results involving the multi summation expressions and ratio of infinite products by using well known m dissections of the power series.***

***Key words: ratio's of infinite products, Bailey pair's, Bailey lemma, Rogers-Ramanujan multi sum identities.***

## 1. Introduction

The m-dissection of the power series $P=\sum_{n=0}^{\infty} a_n q^n$ is the representation of P as $P=P_0+P_1 \dots\dots\dots + P_{m-1}$, where $P_k=\sum_{n=0}^{\infty} a_{mn+k} q^{mn+k}$ Andrews[2] and Hirschhorn[7] have given the 2 dissection and 5 dissection of the continued fraction C(q) and C(q)$^{-1}$. Lewis et al[10] obtained a conjecture of Hirschhorn on 4 dissection of Ramanujan's Continued Fraction. Denis et al[9] gave equivalent continued fraction representations for ratio's of infinite products.

$$\begin{aligned} S(q) &= \frac{(q^3,q^5;q^8)_\infty}{(q,q^7;q^8)_\infty} \\ &= 1+\frac{q+q^2}{1+}\frac{q^4}{1+}\frac{q^3+q^5}{1+}\frac{q^8}{1+}\frac{q^5+q^{10}}{1+}\frac{q^{12}}{1+\cdots} \\ &= 1+\frac{q+q^2}{1-q+}\frac{q+q^4}{1-q+}\frac{q+q^6}{1-q+\cdots} \\ &= 1+q+\frac{q^2}{1+q^3+}\frac{q^4}{1+q^5+}\frac{q^5}{1+q^7+\cdots} \end{aligned}$$

The Bailey chain is a well-known and frequently used technique in the theory of partitions. It arose from W.N. Bailey's realization [11] that the Rogers-Ramanujan identities could be derived from the simple observation that if $\{\alpha_0,\alpha_1,\dots\}$ and $\{\delta_0,\delta_1,\dots\}$ are sequences that satisfy.

$$\beta_k=\sum_{r=0}^{k}\alpha_r u_{k-r} v_{k+r} \quad \text{and} \quad \gamma_k=\sum_{r=k}^{\infty}\delta_r u_{r-k} v_{r+k}, \text{ then}$$

$$\sum_{k=0}^{\infty}\alpha_k\gamma_k=\sum_{k=0}^{\infty}\beta_k\delta_k,$$

provided all infinite sums converge uniformly. L.J. Slater used his idea to produce her list of 130 identities of the Rogers-Ramanujan type [5,6].

A pair of sequences $(\alpha_n(a,q),\ \beta_n(a,q))$ is called a Bailey pair with parameters $(a,q)$ if

$$\beta_n(a,q) = \sum_{r=0}^{n} \frac{\alpha_r(a,q)}{(q;q)_{n-r}(aq;q)_{n+r}} \qquad \text{for } n \geq 0.$$

The unit Bailey's pair [3,4]

$$\beta_n(a,q) = \begin{cases} 1 & if\ n = 0 \\ 0 & if\ n > 0, \end{cases} \qquad \alpha_n(a,q) = \frac{(a;q)_n(1-aq^{2n})}{(q;q)_n(1-a)}(-1)^n q^{(n^2-n)/2}$$

In 2001 D. Bressoud established some interested theorem involving change of base in Bailey pairs, which are

(1.1) Suppose that $(\alpha_n(a,q), \beta_n(a,q))$ is a Bailey pair with parameteerrs $(a,q)$ Then $(\alpha'_n(a,q), \beta'_n(a,q))$ is a Bailey pair with parameters $(a,q)$

$$\alpha'_r(a,q) = a^r q^{r^2} \alpha_r(a,q),$$

$$\beta'_n(a,q) = \sum_{k=0}^{n} \frac{a^k q^{k^2}}{(q;q)_{n-k}} \beta_k(a,q).$$

(1.2) Suppose that $(\alpha_n(a,q), \beta_n(a,q))$ is a Bailey pair with parameteerrs $(a,q)$ Then $(\alpha'_n(a,q), \beta'_n(a,q))$ is a Bailey pair with parameters $(a^2, q^2)$

$$\alpha'_r(a,q) = \alpha_r(a^2,q^2),$$

$$\beta'_n(a,q) = \sum_{k=0}^{n} \frac{(-aq;q)_{2k}}{(q^2;q^2)_{n-k}} q^{n-k} \beta_k(a^2,q^2),$$

(1.3) Suppose that $(\alpha_n(a,q), \beta_n(a,q))$ is a Bailey pair with parameteerrs $(a,q)$ Then $(\alpha'_n(a,q), \beta'_n(a,q))$ is a Bailey pair with parameters $(a^3, q^3)$

$$\alpha'_r(a,q) = a^r q^{r^2} \alpha_r(a,q),$$

$$\beta'_n(a,q) = \frac{1}{(a^3q^3;q^3)_{2n}} \sum_{k=0}^{n} \frac{(aq;q)_{3n-k} a^k q^{k^2}}{(q^3;q^3)_{n-k}} \beta_k(a,q).$$

(1.4) Suppose that $(\alpha_n(a,q), \beta_n(a,q))$ is a Bailey pair with parameteerrs $(a,q)$ Then $(\alpha'_n(a,q), \beta'_n(a,q))$ is a Bailey pair with parameters $(a,q)$

$$\alpha'_r(a,q) = a^{-r} q^{-r^2} \alpha_r(a^3,q^3),$$

$$\beta'_n(a,q) = \frac{1}{(aq;q)_{2n}} \sum_{k=0}^{n} \frac{(aq^{2n+1};q^{-1})_{3k}(a^3q^3;q^3)_{2(n-k)}}{(q^3;q^3)_k} (-1)^k q^{3\binom{k}{2}-n^2} a^{-n} \beta_{n-k}(a^3,q^3).$$

Here an attempt has been made to obtained certain results involving the multi summation expressions and ratio of infinite products by using well known m dissections of the power series.

## 2. Notation

Suppose that $|q| < 1$, where q is non-zero complex number, this condition ensures that all the infinite products that we will converge. We will use the notation,

(2.1) $\qquad (z;q)_\infty = \prod_{n=0}^{\infty} (1 - zq^n),$

(2.2) $\qquad [z;q]_\infty = (z;q)_\infty (z^{-1}q;q)_\infty$, (for $z \neq 0$) and often we write

(2.3) $\qquad [z_1, z_2, \ldots \ldots \ldots z_n; q]_\infty = [z_1;q]_\infty [z_2;q]_\infty \ldots \ldots \ldots [z_n;q]_\infty$ ,

The following fact can be easily verified;

(2.4) $\qquad [z^{-1};q]_\infty = -z^{-1}[z;q]_\infty = [zq;q]_\infty$ , (2.5) $[z, zq; q^2]_\infty = [z; q]_\infty$ ,

(2.6) $[z, -z; q]_\infty = [z^2; q^2]_\infty$ ,

(2.7) $[z^{-1}q; q]_\infty = [z; q]_\infty$ ,

(2.8) $[-1; q]_\infty\, [q; q^2]_\infty = 2$ .

We have the following general relations;
Suppose $a_1, a_2, \dots\dots\dots, a_n; b_1, b_2, \dots\dots\dots, b_n \in C \setminus\{0\}$ satisfy
i) $a_i \neq q^n a_j$ for $i \neq j$ and $n \in z$,
ii) $a_1, a_2, \dots\dots\dots, a_n = b_1, b_2, \dots\dots\dots, b_n$,
Then

(2.9) $$\sum_{i=1}^{n} \frac{\prod_{j=1}^{n}[a_i^{-1}b_j;q]_\infty}{\prod_{j=1,j\neq i}^{n}[a_i^{-1}a_j;q]_\infty} = 0$$

This theorem appears without proof as given by Slater [4] and with a proof as given by Lewis [8].
Also, we have the following well known Rogers- Ramanujan identity.

(2.10) $$\sum_{s_1,s_2\geq 0} \frac{q^{3s_1^2+s_2^2+s_2}(q;q)_{3s_1-s_2}}{(q^3;q^3)_{2s_1}(q^3;q^3)_{s_1-s_2}(q^2;q^2)_{s_2}} = \frac{(q^{10},q^4,q^6;q^{10})_\infty}{(q^3;q^3)_\infty}$$

[D.Bressoud,1;p452]

## 3. Main Result:

(3.1) $$\sum_{s_1,s_2\geq 0} \frac{q^{3s_1^2+s_2^2+s_2}(q;q)_{3s_1-s_2}}{(q^3;q^3)_{2s_1}(q^3;q^3)_{s_1-s_2}(q^2;q^2)_{s_2}} = \frac{(q^3,q^7,q^{10};q^{10})_\infty}{(q^3;q^3)_\infty}\left[\frac{(q,q^9,q^{11},q^{19};q^{20})_\infty(q^8,q^{32};q^{40})_\infty}{(q^{10},q^{10};q^{20})_\infty(q^2,q^6q^{20},q^{20}q^{34},q^{38};q^{40})_\infty}\right.$$

$$+\frac{q(q,q^9,q^{11}q^{19};q^{20})_\infty(q^{12},q^{28};q^{40})_\infty}{(q^{10},q^{10};q^{20})_\infty(q^2,q^{14}q^{20},q^{20}q^{26},q^{38};q^{40})_\infty}$$

$$+\frac{q^2(q^4,q^6,q^8,q^{12},q^{14},q^{16};q^{20})_\infty(q^2,q^{10}q^{30},q^{38};q^{40})_\infty}{(q,q^5,q^{15},q^{19};q^{20})_\infty(q^6,q^8,q^{14},q^{20},q^{20},q^{26},q^{32},q^{34};q^{40})_\infty}$$

$$\left.-\frac{q^2(q^5,q^9,q^{11},q^{15};q^{20})_\infty(q^4,q^{36},q^{36};q^{40})_\infty}{(q^2,q^{18};q^{20})_\infty(q^6,q^{14},q^{20},q^{20},q^{26},q^{34},q^{44};q^{40})_\infty}\right]$$

(3.2) $$\frac{(q^3,q^7,q^{10};q^{10})_\infty}{(q^3;q^3)_\infty} = \left[\frac{(q,q^9,q^{11},q^{19};q^{20})_\infty(q^{14},q^{26};q^{40})_\infty}{(q^{10},q^{10};q^{20})_\infty(q^2,q^{12},q^{20},q^{20}q^{28},q^{38};q^{40})_\infty}\right.$$

$$+\frac{q(q,q^9,q^{11}q^{19};q^{20})_\infty(q^6,q^{34};q^{40})_\infty}{(q^{10},q^{10};q^{20})_\infty(q^2,q^8,q^{20},q^{20},q^{32},q^{38};q^{40})_\infty}+\frac{q^2(q^4,q^6,q^8,q^{12},q^{14},q^{16};q^{20})_\infty(q^2,q^{10}q^{30},q^{38};q^{40})_\infty}{(q,q^5,q^{15},q^{19};q^{20})_\infty(q^6,q^8,q^{14},q^{20},q^{20},q^{26},q^{32},q^{34};q^{40})_\infty}$$

$$\left.+\frac{q^2(q^5,q^9,q^{11},q^{15};q^{20})_\infty(q^4,q^{36},q^{36};q^{40})_\infty}{(q^2,q^{18};q^{20})_\infty(q^8,q^{12},q^{20},q^{20},q^{28},q^{32},q^{44};q^{40})_\infty}\right]\sum_{s_1,s_2\geq 0} \frac{q^{3s_1^2+s_2^2+s_2}(q;q)_{3s_1-s_2}}{(q^3;q^3)_{2s_1}(q^3;q^3)_{s_1-s_2}(q^2;q^2)_{s_2}}$$

## 4. Proof

Considering A(q) = $\frac{(q^4,q^6;q^{10})_\infty}{(q^3,q^7;q^{10})_\infty}$, and A(q) = $\frac{(q^4,q^6;q^{10})_\infty}{(q^3,q^7;q^{10})_\infty}$ can be written as A(q) = $\frac{[q^4,q^{14};q^{20}]_\infty}{[q^3,q^{13};q^{20}]_\infty}$, by using (2.2)

Now, setting $(a_1,a_2,a_3,a_4;b_1,b_2,b_3,b_4)$ = (1,-1, $q^3,q^{13};q^4,q^{14},-q^{-1},q^{-1}$) and taking $q^{20}$ for q in (2.9),

(4.1) $$\frac{[q^4,q^{14},-q^{-1},q^{-1};q^{20}]_\infty}{[-1,q^3,q^{13};q^{20}]_\infty}+\frac{[-q^4,-q^{14},-q^{-1},q^{-1};q^{20}]_\infty}{[-1,-q^3,-q^{13};q^{20}]_\infty}+\frac{[q,q^{11},-q^{-4},q^{-4};q^{20}]_\infty}{[q^{-3},-q^{-3},q^{10};q^{20}]_\infty}+\frac{[q^{-9},q,-q^{-14},q^{-14};q^{20}]_\infty}{[q^{-13},-q^{-13},q^{-10};q^{20}]_\infty}=0$$

By using (2.6) and (2.8) in (4.1)

$$\frac{[q^4,q^{14};q^{20}]_\infty}{[q^3,q^{13};q^{20}]_\infty}+\frac{[-q^4,-q^{14};q^{20}]_\infty}{[-q^3,-q^{13};q^{20}]_\infty}=\frac{2}{[q^2,q^{20};q^{40}]_\infty(-q^{-2})}\left[-\frac{[q,q^{11};q^{20}]_\infty[q^{-8};q^{40}]_\infty}{[q^{10};q^{20}]_\infty[q^{-6};q^{40}]_\infty}-\frac{[q,q^{-9};q^{20}]_\infty[q^{-28};q^{40}]_\infty}{[q^{-10};q^{20}]_\infty[q^{-26};q^{40}]_\infty}\right]$$

By applying (2.4),

$$\frac{[q^4,q^{14};q^{20}]_\infty}{[q^3,q^{13};q^{20}]_\infty}+\frac{[-q^4,-q^{14};q^{20}]_\infty}{[-q^3,-q^{13};q^{20}]_\infty}=\frac{2}{[q^2,q^{20};q^{40}]_\infty(-q^{-2})}\left[-\frac{[q,q^{11};q^{20}]_\infty[q^{8};q^{40}]_\infty}{[q^{10};q^{20}]_\infty[q^{6};q^{40}]_\infty}(q^{-2})-\frac{[q,q^{9};q^{20}]_\infty[q^{28};q^{40}]_\infty}{[q^{10};q^{20}]_\infty[q^{26};q^{40}]_\infty}(q^{-1})\right]$$

(4.2) $$A(q)+A'(q)=\frac{2[q,q^{11};q^{20}]_\infty[q^8;q^{40}]_\infty}{[q^{10};q^{20}]_\infty[q^2,q^6,q^{20};q^{40}]_\infty}+\frac{2q[q,q^9;q^{20}]_\infty[q^{28};q^{40}]_\infty}{[q^{10};q^{20}]_\infty[q^2,q^{20},q^{26};q^{40}]_\infty}$$

(4.3) $$\alpha_1(q)=\frac{1}{2}[A(q)+A'(q)]$$

$$=\frac{[q,q^{11};q^{20}]_\infty[q^8;q^{40}]_\infty}{[q^{10};q^{20}]_\infty[q^2,q^6,q^{20};q^{40}]_\infty}+\frac{q[q,q^9;q^{20}]_\infty[q^{28};q^{40}]_\infty}{[q^{10};q^{20}]_\infty[q^2,q^{20},q^{26};q^{40}]_\infty}$$

Again, setting $(a_1, a_2, a_3, a_4; b_1, b_2, b_3, b_4)$ = (1,-1, $q^{18}, -q^{22}; q^4, q^6, -q^{17}, -q^{13}$) and taking $q^{20}$ for q in (2.9),

$$\frac{[q^4,q^6,-q^{17},-q^{13};q^{20}]_\infty}{[-1,q^{18},-q^{22};q^{20}]_\infty} + \frac{[-q^4,-q^6,q^{17},q^{13};q^{20}]_\infty}{[-1,-q^{18},q^{22};q^{20}]_\infty} + \frac{[q^{-14},q^{-12},-q^{-1},-q^{-5};q^{20}]_\infty}{[q^{-18},-q^{-18},-q^4;q^{20}]_\infty}$$

$$+ \frac{[-q^{-18},-q^{-16},q^{-5},q^{-9};q^{20}]_\infty}{[q^{-22},-q^{-22},-q^{-4};q^{20}]_\infty} = 0$$

By applying (2.4) and (2.7),

$$\frac{q^2}{[-1,q^2,-q^2;q^{20}]_\infty}[[q^4,q^{14},-q^3,-q^{13};q^{20}]_\infty - [-q^4,-q^{14},q^3,q^{13};q^{20}]_\infty]$$

$$- \frac{q^4[q^{14},q^{12},-q,-q^5;q^{20}]_\infty}{[q^{18},-q^{18},-q^4;q^{20}]_\infty} - \frac{[-q^{16},-q^{18},q^5,q^9;q^{20}]_\infty}{[q^{22},-q^{22},-q^4;q^{20}]_\infty} = 0$$

By using (2.6)

$$[[q^4,q^{14},-q^3,-q^{13};q^{20}]_\infty - [-q^4,-q^{14},q^3,q^{13};q^{20}]_\infty]$$

$$= \frac{2[q^4;q^{40}]_\infty}{q^2[q^{20};q^{40}]_\infty}\left[\frac{q^4[q^4,q^{12},q^{14};q^{20}]_\infty[q^2,q^{10};q^{40}]_\infty}{[q,q^5;q^{20}]_\infty[q^8,q^{36};q^{40}]_\infty} + \frac{[q^4,q^5,q^9;q^{20}]_\infty[q^{32},q^{36};q^{40}]_\infty}{[q^{16},q^{18};q^{20}]_\infty[,q^8,q^{44};q^{40}]_\infty}\right]$$

Dividing by $[-q^3,q^3,q^{13},-q^{13};q^{20}]_\infty$ and using (2.6),

$$\frac{[q^4,q^{14};q^{20}]_\infty}{[q^3,q^{13};q^{20}]_\infty} - \frac{[-q^4,-q^{14};q^{20}]_\infty}{[-q^3,-q^{13};q^{20}]_\infty}$$

$$= \frac{2[q^4;q^{40}]_\infty}{q^2[q^{20};q^{40}]_\infty[q^6,q^{26};q^{40}]_\infty}\left[\frac{q^4[q^4,q^{12},q^{14};q^{20}]_\infty[q^2,q^{10};q^{40}]_\infty}{[q,q^5;q^{20}]_\infty[q^8,q^{36};q^{40}]_\infty} + \frac{[q^4,q^5,q^9;q^{20}]_\infty[q^{32},q^{36};q^{40}]_\infty}{[q^{16},q^{18};q^{20}]_\infty[,q^8,q^{44};q^{40}]_\infty}\right]$$

$$\frac{[q^4,q^{14};q^{20}]_\infty}{[q^3,q^{13};q^{20}]_\infty} - \frac{[-q^4,-q^{14};q^{20}]_\infty}{[-q^3,-q^{13};q^{20}]_\infty}$$

$$= \left[\frac{2q^2[q^4,q^{12},q^{14};q^{20}]_\infty[q^2,q^4,q^{10};q^{40}]_\infty}{[q,q^5;q^{20}]_\infty[q^6,q^8,q^{20},q^{26},q^{36};q^{40}]_\infty} + \frac{2[q^4,q^5,q^9;q^{20}]_\infty[q^4,q^{32},q^{36};q^{40}]_\infty}{q^2[q^{16},q^{18};q^{20}]_\infty[q^6,q^8,q^{20},q^{26},q^{44};q^{40}]_\infty}\right]$$

(4.4) $\quad A(q) - A'(q) = \frac{2q^2[q^4,q^{12},q^{14};q^{20}]_\infty[q^2,q^{10};q^{40}]_\infty}{[q,q^5;q^{20}]_\infty[q^6,q^8,q^{20},q^{26};q^{40}]_\infty} + \frac{2[q^5,q^9;q^{20}]_\infty[q^4,q^{36};q^{40}]_\infty}{q^2[q^{18};q^{20}]_\infty[q^6,q^{20},q^{26},q^{44};q^{40}]_\infty}$

(4.5) $\quad \beta_1(q) = \frac{1}{2}\left[A(q) - A'(q)\right]$

 

$$= \frac{q^2[q^4,q^{12},q^{14};q^{20}]_\infty[q^2,q^{10};q^{40}]_\infty}{[q,q^5;q^{20}]_\infty[q^6,q^8,q^{20},q^{26};q^{40}]_\infty} + \frac{[q^5,q^9;q^{20}]_\infty[q^4,q^{36};q^{40}]_\infty}{q^2[q^{18};q^{20}]_\infty[q^6,q^{20},q^{26},q^{44};q^{40}]_\infty}$$

By adding (4.3) and (4.5),

$$A(q) = \alpha_1(q) + \beta_1(q)$$

(4.6) $$A(q) = \frac{[q,q^{11};q^{20}]_\infty[q^8;q^{40}]_\infty}{[q^{10};q^{20}]_\infty[q^2,q^6,q^{20};q^{40}]_\infty} + \frac{q[q,q^9;q^{20}]_\infty[q^{28};q^{40}]_\infty}{[q^{10};q^{20}]_\infty[q^2,q^{20},q^{26};q^{40}]_\infty}$$

$$+ \frac{q^2[q^4,q^{12},q^{14};q^{20}]_\infty[q^2,q^{10};q^{40}]_\infty}{[q,q^5;q^{20}]_\infty[q^6,q^8,q^{20},q^{26};q^{40}]_\infty} + \frac{[q^5,q^9;q^{20}]_\infty[q^4,q^{36};q^{40}]_\infty}{q^2[q^{18};q^{20}]_\infty[q^6,q^{20},q^{26},q^{44};q^{40}]_\infty}$$

By applying (2.2) in (4.6),

(4.7) $$A(q) = \frac{(q,q^9,q^{11},q^{19};q^{20})_\infty(q^8,q^{32};q^{40})_\infty}{(q^{10},q^{10};q^{20})_\infty(q^2,q^6q^{20},q^{20}q^{34},q^{38};q^{40})_\infty}$$

$$+ \frac{q(q,q^9,q^{11}q^{19};q^{20})_\infty(q^{12},q^{28};q^{40})_\infty}{(q^{10},q^{10};q^{20})_\infty(q^2,q^{14}q^{20},q^{20}q^{26},q^{38};q^{40})_\infty}$$

$$+ \frac{q^2(q^4,q^6,q^8,q^{12},q^{14},q^{16};q^{20})_\infty(q^2,q^{10}q^{30},q^{38};q^{40})_\infty}{(q,q^5,q^{15},q^{19};q^{20})_\infty(q^6,q^8,q^{14},q^{20},q^{20},q^{26},q^{32},q^{34};q^{40})_\infty}$$

$$- \frac{q^2(q^5,q^9,q^{11},q^{15};q^{20})_\infty(q^4,q^{36},q^{36};q^{40})_\infty}{(q^2,q^{18};q^{20})_\infty(q^6,q^{14},q^{20},q^{20},q^{26},q^{34},q^{44};q^{40})_\infty}$$

By taking known identity (2.10)

$$\sum_{s_1,s_2\geq 0} \frac{q^{3s_1^2+s_2^2+s_2}(q;q)_{3s_1-s_2}}{(q^3;q^3)_{2s_1}(q^3;q^3)_{s_1-s_2}(q^2;q^2)_{s_2}} = \frac{(q^{10},q^4,q^6;q^{10})_\infty}{(q^3;q^3)_\infty}$$

$$\sum_{s_1,s_2\geq 0} \frac{q^{3s_1^2+s_2^2+s_2}(q;q)_{3s_1-s_2}}{(q^3;q^3)_{2s_1}(q^3;q^3)_{s_1-s_2}(q^2;q^2)_{s_2}} = \frac{(q^3,q^7,q^{10};q^{10})_\infty}{(q^3;q^3)_\infty}\frac{(q^4,q^6;q^{10})_\infty}{(q^3,q^7;q^{10})_\infty}$$

(4.8) $$\sum_{s_1,s_2\geq 0} \frac{q^{3s_1^2+s_2^2+s_2}(q;q)_{3s_1-s_2}}{(q^3;q^3)_{2s_1}(q^3;q^3)_{s_1-s_2}(q^2;q^2)_{s_2}} = \frac{(q^3,q^7,q^{10};q^{10})_\infty}{(q^3;q^3)_\infty} A(q)$$

Now, by putting the value of A(q) from (4.7) in (4.8),

(4.9) $$\sum_{s_1,s_2\geq 0} \frac{q^{3s_1^2+s_2^2+s_2}(q;q)_{3s_1-s_2}}{(q^3;q^3)_{2s_1}(q^3;q^3)_{s_1-s_2}(q^2;q^2)_{s_2}} = \frac{(q^3,q^7,q^{10};q^{10})_\infty}{(q^3;q^3)_\infty}\left[\frac{(q,q^9,q^{11},q^{19};q^{20})_\infty(q^8,q^{32};q^{40})_\infty}{(q^{10},q^{10};q^{20})_\infty(q^2,q^6q^{20},q^{20}q^{34},q^{38};q^{40})_\infty}\right.$$

$$+\frac{q\left(q,q^{9},q^{11}q^{19};q^{20}\right)_{\infty}\left(q^{12},q^{28};q^{40}\right)_{\infty}}{(q^{10},q^{10};q^{20})_{\infty}(q^{2},q^{14}q^{20},q^{20}q^{26},q^{38};q^{40})_{\infty}}$$

$$+\frac{q^{2}\left(q^{4},q^{6},q^{8},q^{12},q^{14},q^{16};q^{20}\right)_{\infty}\left(q^{2},q^{10}q^{30},q^{38};q^{40}\right)_{\infty}}{(q,q^{5},q^{15},q^{19};q^{20})_{\infty}(q^{6},q^{8},q^{14},q^{20},q^{20},q^{26},q^{32},q^{34};q^{40})_{\infty}}$$

$$-\frac{q^{2}\left(q^{5},q^{9},q^{11},q^{15};q^{20}\right)_{\infty}\left(q^{4},q^{36},q^{36};q^{40}\right)_{\infty}}{(q^{2},q^{18};q^{20})_{\infty}(q^{6},q^{14},q^{20},q^{20},q^{26},q^{34},q^{44};q^{40})_{\infty}}\Bigg]$$

This is the main result (3.1).

Again, considering (4.2)

$$\mathrm{A(q)} + \mathrm{A}'(\mathrm{q}) = \frac{2\left[q,q^{11};q^{20}\right]_{\infty}\left[q^{8};q^{40}\right]_{\infty}}{[q^{10};q^{20}]_{\infty}[q^{2},q^{6},q^{20};q^{40}]_{\infty}} + \frac{2q\left[q,q^{9};q^{20}\right]_{\infty}\left[q^{28};q^{40}\right]_{\infty}}{[q^{10};q^{20}]_{\infty}[q^{2},q^{20},q^{26};q^{40}]_{\infty}}$$

Multiplying (4.2) by $\frac{\left[q^{6},q^{26};q^{40}\right]_{\infty}}{[q^{8},q^{28};q^{40}]_{\infty}}$ ,

(4.10) $$\frac{\left[-q^{3},-q^{13};q^{20}\right]_{\infty}}{[-q^{4},-q^{14};q^{20}]_{\infty}}+\frac{\left[q^{3},q^{13};q^{20}\right]_{\infty}}{[q^{4},q^{14};q^{20}]_{\infty}} = \frac{2\left[q,q^{11};q^{20}\right]_{\infty}\left[q^{26};q^{40}\right]_{\infty}}{[q^{10};q^{20}]_{\infty}[q^{2},q^{20},q^{28};q^{40}]_{\infty}} + \frac{2q\left[q,q^{9};q^{20}\right]_{\infty}\left[q^{6};q^{40}\right]_{\infty}}{[q^{10};q^{20}]_{\infty}[q^{2},q^{8},q^{20};q^{40}]_{\infty}}$$

(4.11) $$\alpha_{2}(\mathrm{q}) = \frac{1}{2}\left[\mathrm{A}'(\mathrm{q})^{-1} + \mathrm{A}(\mathrm{q})^{-1}\right]$$

$$= \frac{\left[q,q^{11};q^{20}\right]_{\infty}\left[q^{26};q^{40}\right]_{\infty}}{[q^{10};q^{20}]_{\infty}[q^{2},q^{20},q^{28};q^{40}]_{\infty}} + \frac{q\left[q,q^{9};q^{20}\right]_{\infty}\left[q^{6};q^{40}\right]_{\infty}}{[q^{10};q^{20}]_{\infty}[q^{2},q^{8},q^{20};q^{40}]_{\infty}}$$

Again, multiplying (4.4) by $\frac{\left[q^{6},q^{26};q^{40}\right]_{\infty}}{[q^{8},q^{28};q^{40}]_{\infty}}$,

(4.12) $$\mathrm{A(q)}^{-1} - \mathrm{A}'(\mathrm{q})^{-1} = -\frac{2q^{2}\left[q^{4},q^{12},q^{14};q^{20}\right]_{\infty}\left[q^{2},q^{10};q^{40}\right]_{\infty}}{[q,q^{5};q^{20}]_{\infty}[q^{8},q^{8},q^{20},q^{28};q^{40}]_{\infty}} - \frac{2\left[q^{5},q^{9};q^{20}\right]_{\infty}\left[q^{4},q^{36};q^{40}\right]_{\infty}}{q^{2}[q^{18};q^{20}]_{\infty}[q^{8},q^{20},q^{28},q^{44};q^{40}]_{\infty}}$$

(4.13) $$\beta_{2}(\mathrm{q}) = \frac{1}{2}\left[\mathrm{A(q)}^{-1} - \mathrm{A}'(\mathrm{q})^{-1}\right]$$

$$= -\frac{q^{2}\left[q^{4},q^{12},q^{14};q^{20}\right]_{\infty}\left[q^{2},q^{10};q^{40}\right]_{\infty}}{[q,q^{5};q^{20}]_{\infty}[q^{8},q^{8},q^{20},q^{28};q^{40}]_{\infty}} - \frac{\left[q^{5},q^{9};q^{20}\right]_{\infty}\left[q^{4},q^{36};q^{40}\right]_{\infty}}{q^{2}[q^{18};q^{20}]_{\infty}[q^{8},q^{20},q^{28},q^{44};q^{40}]_{\infty}}$$

By adding (4.11) and (4.13),

$$\mathrm{A(q)}^{-1} = \alpha_2(q) + \beta_2(q)$$

(4.14) $$\mathrm{A(q)}^{-1} = \frac{[q,q^{11};q^{20}]_\infty [q^{26};q^{40}]_\infty}{[q^{10};q^{20}]_\infty [q^2,q^{20},q^{28};q^{40}]_\infty} + \frac{q[q,q^9;q^{20}]_\infty [q^6;q^{40}]_\infty}{[q^{10};q^{20}]_\infty [q^2,q^8,q^{20};q^{40}]_\infty}$$

$$-\frac{q^2[q^4,q^{12},q^{14};q^{20}]_\infty [q^2,q^{10};q^{40}]_\infty}{[q,q^5;q^{20}]_\infty [q^8,q^8,q^{20},q^{28};q^{40}]_\infty} - \frac{[q^5,q^9;q^{20}]_\infty [q^4,q^{36};q^{40}]_\infty}{q^2[q^{18};q^{20}]_\infty [q^8,q^{20},q^{28},q^{44};q^{40}]_\infty}$$

By applying (2.2) in (4.14),

(4.15) $$\mathrm{A(q)}^{-1} = \frac{(q,q^9,q^{11},q^{19};q^{20})_\infty (q^{14},q^{26};q^{40})_\infty}{(q^{10},q^{10};q^{20})_\infty (q^2,q^{12},q^{20},q^{20}q^{28},q^{38};q^{40})_\infty}$$

$$+\frac{q(q,q^9,q^{11}q^{19};q^{20})_\infty (q^6,q^{34};q^{40})_\infty}{(q^{10},q^{10};q^{20})_\infty (q^2,q^8,q^{20},q^{20},q^{32},q^{38};q^{40})_\infty}$$

$$+\frac{q^2(q^4,q^6,q^8,q^{12},q^{14},q^{16};q^{20})_\infty (q^2,q^{10}q^{30},q^{38};q^{40})_\infty}{(q,q^5,q^{15},q^{19};q^{20})_\infty (q^6,q^8,q^{14},q^{20},q^{20},q^{26},q^{32},q^{34};q^{40})_\infty}$$

$$+\frac{q^2(q^5,q^9,q^{11},q^{15};q^{20})_\infty (q^4,q^{36},q^{36};q^{40})_\infty}{(q^2,q^{18};q^{20})_\infty (q^8,q^{12},q^{20},q^{20},q^{28},q^{32},q^{44};q^{40})_\infty}$$

Now, from (4.8), can be written as

(4.16) $$\mathrm{A(q)}^{-1} \sum_{s_1,s_2\geq 0} \frac{q^{3s_1^2+s_2^2+s_2}(q;q)_{3s_1-s_2}}{(q^3;q^3)_{2s_1}(q^3;q^3)_{s_1-s_2}(q^2;q^2)_{s_2}} = \frac{(q^3,q^7,q^{10};q^{10})_\infty}{(q^3;q^3)_\infty}$$

By using (4.15) in (4.16)

(4.17) $$\frac{(q^3,q^7,q^{10};q^{10})_\infty}{(q^3;q^3)_\infty} = \Bigg[\frac{(q,q^9,q^{11},q^{19};q^{20})_\infty (q^{14},q^{26};q^{40})_\infty}{(q^{10},q^{10};q^{20})_\infty (q^2,q^{12},q^{20},q^{20}q^{28},q^{38};q^{40})_\infty}$$

$$+\frac{q(q,q^9,q^{11}q^{19};q^{20})_\infty (q^6,q^{34};q^{40})_\infty}{(q^{10},q^{10};q^{20})_\infty (q^2,q^8,q^{20},q^{20},q^{32},q^{38};q^{40})_\infty} + \frac{q^2(q^4,q^6,q^8,q^{12},q^{14},q^{16};q^{20})_\infty (q^2,q^{10}q^{30},q^{38};q^{40})_\infty}{(q,q^5,q^{15},q^{19};q^{20})_\infty (q^6,q^8,q^{14},q^{20},q^{20},q^{26},q^{32},q^{34};q^{40})_\infty}$$

$$+\frac{q^2(q^5,q^9,q^{11},q^{15};q^{20})_\infty (q^4,q^{36},q^{36};q^{40})_\infty}{(q^2,q^{18};q^{20})_\infty (q^8,q^{12},q^{20},q^{20},q^{28},q^{32},q^{44};q^{40})_\infty}\Bigg] \sum_{s_1,s_2\geq 0} \frac{q^{3s_1^2+s_2^2+s_2}(q;q)_{3s_1-s_2}}{(q^3;q^3)_{2s_1}(q^3;q^3)_{s_1-s_2}(q^2;q^2)_{s_2}}$$

This is the main result (3.2).

 

## References


1. D. Bressoud, M. E. H. Ismail, and D. Stanton, Change of Base in Bailey Pairs, *The Ramanujan Journal*, **4,** (2000), 435-453.
2. G. Andrews, Multiple series Rogers-Ramanujan type identities, *Pac. J. Math.* **114** (1984), 267-283.
3. G. E. Andrews, Ramanujan's lost notebook III-The Rogers-Ramanujans Continued Fraction, *Adv. Math*,**41** (1981), 186-208.
4. L. J. Slater, Generalized Hyper geometric Function, camb. Univ. Press, Cambridge, (1996).
5. L. J. Slater, A new proof of Roger's transformations of infinite series, *Proc. London Math. Soc. (2)* **53,** (1951), 460-475.
6. L. J. Slater, Further identities of the Rogers-Ramanujan type, *Proc. London Math. Soc. (2)* **54,** (1952), 147-167.
7. M. D. Hirschhorn, on the the expansion of Ramanujan's Continued Fraction, *The Ramanujan's Journal*, **2** (1998), 521-527.
8. R. P. Lewis, On the Ranks of Partitions Modulo9, *Bull. London Math Soc.*,23 (1991), 417-421.
9. R. Y. Denis, S. N. Singh and N. A. Herbert, On certain Bilateral q-series and Ramanujan's Continued Fraction, *South East Asia J. Math. &. Math Sc.*, **3(1)**, (2004),49-61.
10. Richard Lewis and Zhi-Guo Liu, A Conjecture of Hirschhorn on the 4-dissection of Ramanujan's Continued Fraction, *The Ramanujan's Journal,* **4** (2001), 347-352.
11. W. N. Bailey, Identities of the Rogers-Ramanujan type, *Proc. London Math. Soc. (2)* **50,** (1949), 1-10.