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CHAOTIC BEHAVIOR IN DISCRETE CHEN SYSTEM WITH FRACTIONAL ORDER

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Abstract - Guanrong Chen and Ueta found a simple three dimensional autonomous system in 1999 and this system is not topologically similar to Lorenz's system as well as a double scroll chaotic attractor. This work examins the dynamical nature of the fractional discrete time Chen system with order between zero and one. The ordinary differential equation of the Chen system is described by the following form

$$\dot{x} = ay(\tau) - ax(\tau); \dot{y} = cx(\tau) - ax(\tau) - x(\tau)z(\tau) + cy(\tau); \dot{z} = x(\tau)y(\tau) - bz(\tau)$$

Here a, b, c ***assume non-zero positive values. Applying a discretization process, its discrete version of FO Chen system is obtained. Fixed points are obtained and using Routh-Hurwitz criteria, the stability of the system is analysed. Numerical verification are presented for the analysis of dynamical properties of the system. Time series and phase orbit diagrams for the discrete time FO Chen system are constructed. Numerically solving the system of discrete fractional order 3-D Chen system, chaotic nature is explained. For the discrete time FO system, chaotic attractor exists for different parameter values.***

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1. INTRODUCTION

Fractional calculus is a generalization of the ordinary differentiation and integration to arbitrary non-integer order. The subject is as old as the differential calculus and goes back to times when Leibniz and Newton invented differential calculus. The idea of fractional calculus and fractional order differential equations and inclusions have been a subject of interest not only among mathematicians but also among physicists and engineers. The main reason for using integer-order models was the absence of solution methods for fractional differential equations.

At present, there are many methods for approximation of the fractional derivative and integral and fractional calculus can be used in wide areas of applications. In 21st century, many researchers have been attracted towards FODE and find numerous applications in science and technology. In addition, FODEs play vital role and is essential in most biological models. The Basic Caputo definition of FO and its properties of differentiation and Integration can be found in [1, 4].

Definition 1 [5]Let $\alpha \in R^+$ is the fractional order and $g(u), u > 0$ is a function, then the fractional integral is defined by

$$I^\alpha g(u) = \int_0^u \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} g(s)ds$$

also $\gamma \in (m-1, m)$ is the fractional order and $g(u), u > 0$ is a function, then the fractional derivative is defined by

$$D^\gamma f(u) = I^{m-\gamma} D^\gamma f(u), D = \frac{d}{du}.$$

Moreover, the properties of FO differentiation and Integration are used in fractional calculus. Let $\beta, \gamma \in R^+$, $\alpha \in (0,1)$,

1. $I_\alpha^\beta : L^1 \to L^1$, and if $g(y) \in L^1$, then $I_\alpha^\gamma I_\alpha^\beta g(y) = I_\alpha^{\gamma+\beta} g(y)$.
2. $\lim_{\beta \to m} I_\alpha^\beta g(y) = I_\alpha^m g(y)$ uniformly on $[a,b]$, $m = 1,2,3,...,$ where $I_\alpha^1 g(y) = \int_\alpha^y g(t)dt$.
3. $\lim_{\beta \to 0} g(y) = g(y)$ weakly.
4. If $g(x)$ is absolutely continuous on $[a,b]$, then $\lim_{\alpha \to 1} D_a^\alpha g(x) = \frac{dg(x)}{dx}$.

Our main objective of this work is to examine the some dynamical nature and achieve the chaotic attractor in the discrete FO Chen system with order which lies between zero and one. In addition, using the fractional Routh-Hurwitz criteria analysis, we illustrate the asymptotic stability conditions. The rest of this paper organised as follows: In section

2, we present the description of the discrete FO Chen system which established from (1) by applying a discretization method. In section 3, we determine the fixed points and using the Routh-Hurwitz method to enable the stability analysis of the system (3). Numerically, we observe the dynamical behaviour of the system (3) by plotting time lines and phase diagrams which are displayed in section 4. Also we exhibit the chaotic attractors of discrete FO Chen System. Finally, a brief conclusion of our work is provided in the last section.

2. DESCRIPTION OF THE SYSTEM

Let us consider a simple three dimensional autonomous fractional order chaotic Chen system. This system is not topologically similar to the Lorenz system and this was invented by Chen and Ueta in 1999.

$$\begin{aligned} D^{\alpha}x(\tau) &= ay(\tau)-ax(\tau) \\ D^{\alpha}y(\tau) &= cx(\tau)-ax(\tau)-x(\tau)z(\tau)+cy(\tau) \\ D^{\alpha}z(\tau) &= x(\tau)y(\tau)-bz(\tau) \end{aligned} \tag{1}$$

Here (x, y, z) are set of state variables with time τ and (a,b,c) are non zero positive values and α is the fractional derivative between zero and one. In the following discussion, we show the system exhibits chaotic attractor for $\alpha = 0.99$. The chaotic attractor exists for the parameter set $\{a = 35, b = 3, c = 28\}$, but it may not be chaotic for some other parameter set [9].

Now applying a discretization process, we arrive at the discrete time FODE [6, 10, 11]. Using piecewise constant arguments, we get the discrete time FO Chen's system in the form

$$\begin{aligned} D^{\alpha}x(\tau) &= ay\left(r\left[\frac{\tau}{r}\right]\right)-ax\left(r\left[\frac{\tau}{r}\right]\right) \\ D^{\alpha}y(\tau) &= cx\left(r\left[\frac{\tau}{r}\right]\right)-ax\left(r\left[\frac{\tau}{r}\right]\right)-x\left(r\left[\frac{\tau}{r}\right]\right)z\left(r\left[\frac{\tau}{r}\right]\right)+cy\left(r\left[\frac{\tau}{r}\right]\right) \\ D^{\alpha}z(\tau) &= x\left(r\left[\frac{\tau}{r}\right]\right)y\left(r\left[\frac{\tau}{r}\right]\right)-bz\left(r\left[\frac{\tau}{r}\right]\right) \end{aligned} \tag{2}$$

with initial points $(x(0), y(0), z(0)) = (x_0, y_0, z_0)$. By the proposed discretization method, we obtain

$$\begin{aligned} x(\tau+1) &= x(\tau)+\frac{r^{\alpha}}{\Gamma(1+\alpha)}\big(ay(\tau)-ax(\tau)\big) \\ y(\tau+1) &= y(\tau)+\frac{r^{\alpha}}{\Gamma(1+\alpha)}\big(cx(\tau)-ax(\tau)-x(\tau)z(\tau)+cy(\tau)\big) \\ z(\tau+1) &= z(\tau)+\frac{r^{\alpha}}{\Gamma(1+\alpha)}\big(x(\tau)y(\tau)-bz(\tau)\big). \end{aligned} \tag{3}$$

3. STABILITY AND FIXED POINTS OF THE SYSTEM (3)

This section discusses the stability and asymptotic stability of the system. The system (3) has the following three fixed points: $F_1 = (0,0,0)$, $F_2 = \left(\sqrt{(2c-a)b}, \sqrt{(2c-a)b}, (2c-a)\right)$, and $F_3 = \left(-\sqrt{(2c-a)b}, -\sqrt{(2c-a)b}, (2c-a)\right)$ respectively. Analysis involves considering the Jacobian matrix of the fixed points and obtaining eigen values of the system [2, 3, 7]. We can compute the stability of each fixed point from the determination of the roots of the characteristic equation.

$$J = \begin{pmatrix} 1-as & as & 0 \\ (c-a)s-sz & 1+cs & -sx \\ sy & sx & 1-bs \end{pmatrix}$$

where $s = \dfrac{r^{\alpha}}{\Gamma(1+\alpha)}$. We have the following theorem [8] to investigate the linear stability analysis of fixed points of system (3).

Theorem 1: (Jury Conditions) Consider the characteristic polynomial equation: $\lambda^3+\rho_1\lambda^2+\rho_2\lambda+\rho_3 = 0$ where ρ_1, ρ_2 and ρ_3 are real numbers. Then the necessary and sufficient conditions that all the roots of equation lie in the open disk $\lambda < 1$ are:

$$|\rho_1+\rho_3| < 1+\rho_2, |\rho_1-3\rho_3| < 3-\rho_2, \text{and}$$
$$\rho_3^2+\rho_2-\rho_3\rho_1 < 1$$

Now linearizing system (3) about F_1 yields the following characteristic equation:

$$\phi(\lambda) = \lambda^3+\rho_1\lambda^2+\rho_2\lambda+\rho_3 = 0. \tag{4}$$

where $\rho_1 = (B-A), \rho_2 = (C-AB), \rho_3 = -AC$ and $A = 1-bs, B = s(a-c)-2, C = a^2 - 2ac + s(c-a)+1$. From the Jury test, $|\rho_1+\rho_3| < 1+\rho_2, |\rho_1 - 3\rho_3| < 3-\rho_2$, and $\rho_3^2 + \rho_2 - \rho_3\rho_1 < 1$, then the roots of $\phi(\lambda)$ satisfy $\lambda < 1$ thus F_1 is asymptotically stable.

While linearizing system (3) about F_2 or F_3 yields the following characteristic equation:

$$\psi(\lambda) = \lambda^3 + \rho_1\lambda^2 + \rho_2\lambda + \rho_3 = 0. \tag{5}$$

Where $\rho_1 = -(A+B+C), \rho_2 = BC + D^2 + acs^2 - A(B+C), \rho_3 = asD^2 - ABC - AD^2acs^2$ and $D = \sqrt{b(2c-a)}$. From the jury test, $|\rho_1+\rho_3| < 1+\rho_2, |\rho_1 - 3\rho_3| < 3-\rho_2$, and $\rho_3^2 + \rho_2 - \rho_3\rho_1 < 1$, then the roots of $\psi(\lambda)$ satisfy $\lambda < 1$ thus F_2 or F_3 is asymptotically stable.

4. CHAOTIC ATTRACTORS OF DISCRETE FRACTIONAL CHEN SYSTEM

Numerical simulation results are useful in showing the consistency with the theoretical analysis and exhibits interesting dynamical natures. Now, we provide some numerical illustrations for the qualitative nature of the model (3). Bifurcation theory deals with the change in the qualitative properties of a dynamical system when one or more of the parameters change. In the following numerical simulations, we will observe dynamical behavior of the system (3) by plotting the time lines and phase orbit diagrams between zero and one fractional order. Also we analyse chaotic attractors of the discrete fractional order Chen system visually. The numerical examples exhibit the existence of wide range of chaotic dydmanics using phase plane.

Without loss of generality, we fix the parameter values and assume that $\alpha = 0.99, r = 0.001, a = 36, b = 3, c = 20$. From Figure 1, one can see that the initial values $(x_0 = -0.1, y_0 = 0.5, z_0 = -0.6)$ approaches the stable fixed point $f_2 = (3.4641, 3.4641, 4)$. The eigen values of the fixed point are $\lambda_1 = 0.9799$ and $\lambda_{2,3} = 0.9996 \pm i0.0075$; $|\lambda_1| = 0.9799 < 1$ and $|\lambda_{2,3}| = 0.9996 < 1$ also the parameters satisfying the Jury conditions such that $c_1 + c_3 (= 3.9583) < 1 + c_2 (= 3.9585)$; $|c_1 - 3c_3| (= 0.0416) < 3 - c_2 (= 0.0418)$ and $c_3^2 + c_2 - c_3 c_1 = 0.9999 < 1$. Figure-2 displays various phase orbit diagrams of system (3) for the above parameters and varying the fractional order.

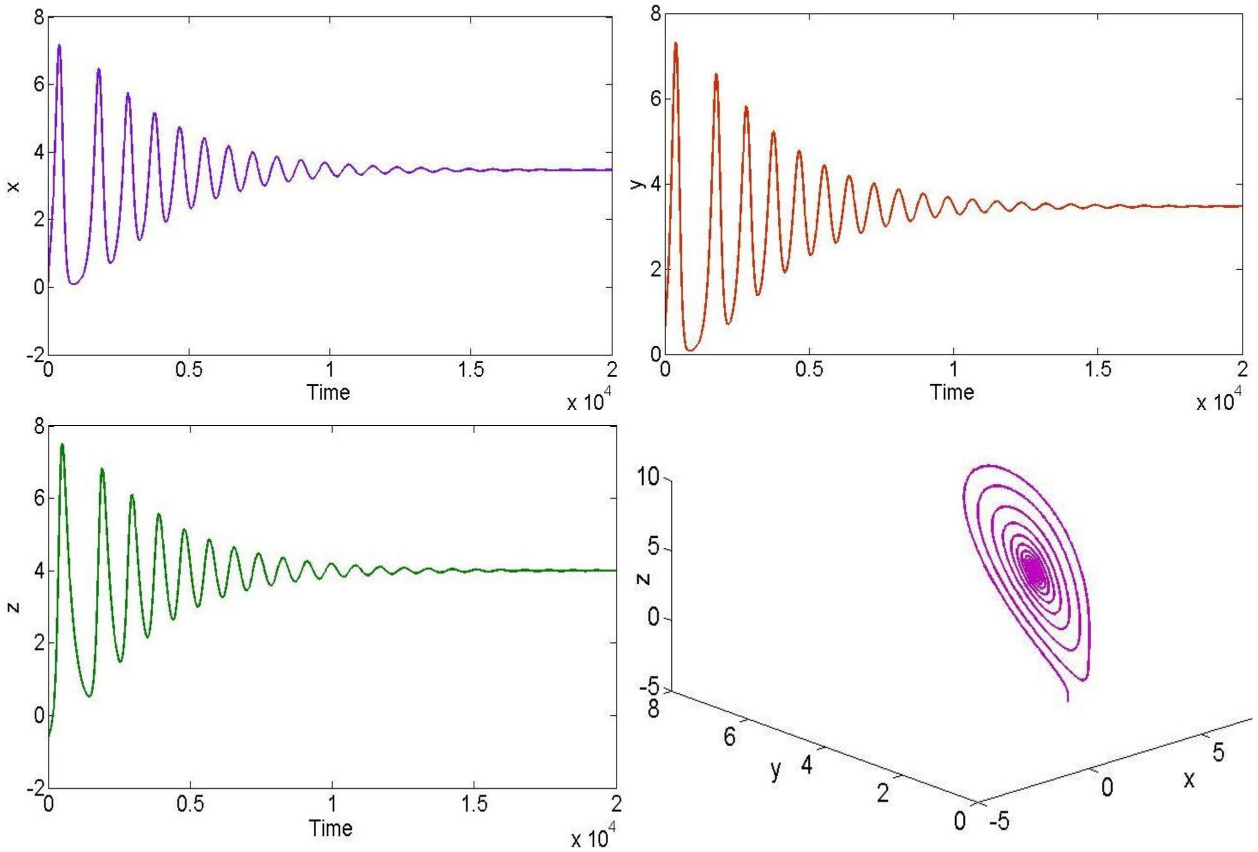

Figure 1: *Time lines and Phase Orbit diagram of interior fixed point f_2 of the discrete fractional order system*

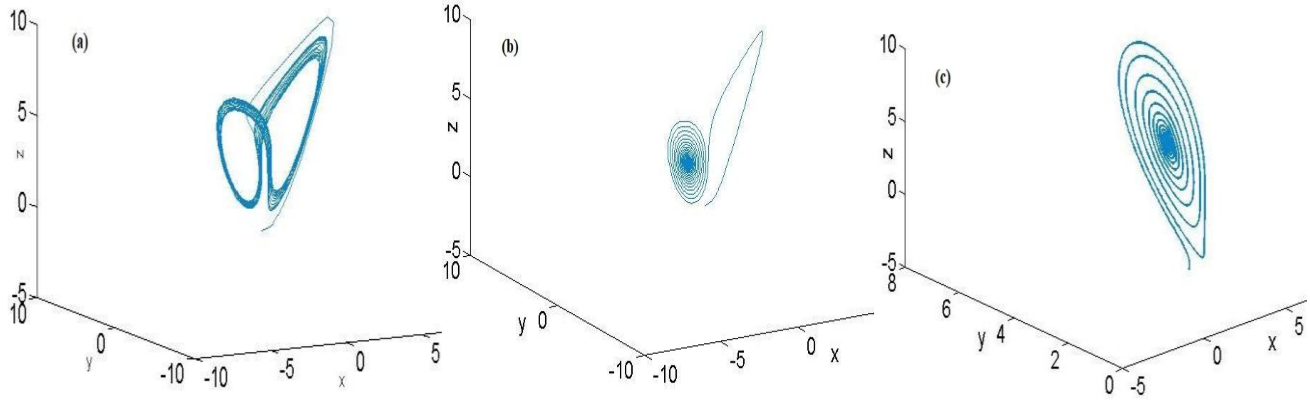

Figure 2: *Different Phase orbit of the System (3) with (a)* $\alpha = 0.5$*, (b)* $\alpha = 0.7$ *, (c)* $\alpha = 0.9$.

Now we fix another set parameters $\alpha = 0.99, r = 0.01, a = 36, b = 3, c = 20$. Figure-3 shows that the trajectory goes to the stable fixed point $f_3 = (-3.4641, -3.4641, 4)$ from the initial points $(x_0 = -0.1, y_0 = 0.5, z_0 = -0.6)$. Here $\lambda_1 = 0.8078$ and $\lambda_{2,3} = 0.9963 \pm i0.0721$ are eigen values of the fixed point f_3; $|\lambda_1| = 0.8078 < 1$ and $|\lambda_{2,3}| = 0.9963 < 1$ and Jury conditions are satisfied as $c_1 + c_3 (= 3.6066) < 1 + c_2 (= 3.6076)$; $|c_1 - 3c_3| (= 0.3822) < 3 - c_2 (= 0.3924)$ and $c_3^2 + c_2 - c_3 c_1 = 0.9999 < 1$. Figure-4 exhibits various phase orbit diagrams of the discrete fractional order Chen system by assuming the same parameter values of Figure-3 and varying the fractional order.

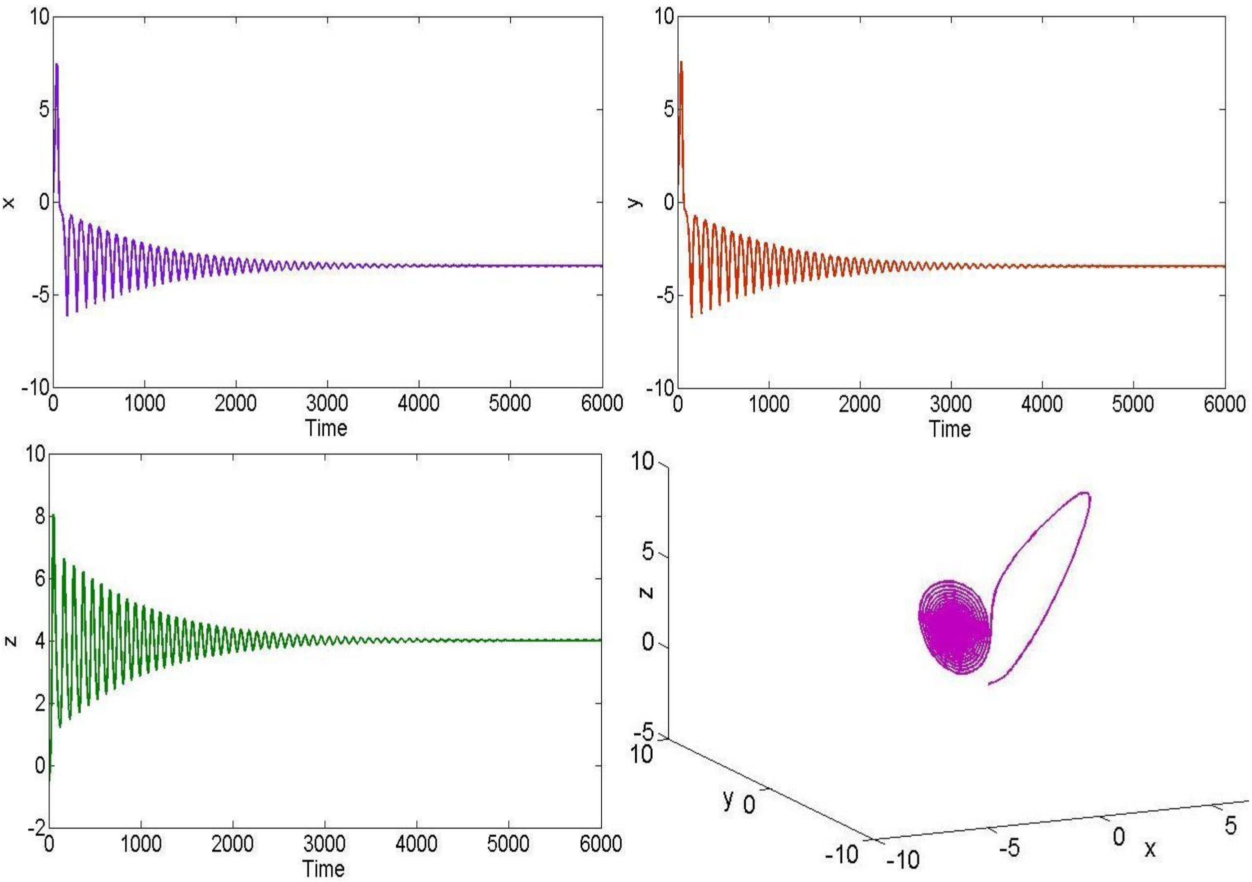

Figure 3: *Time lines and Phase Orbit of interior fixed point* f_3 *of the discrete fractional order system*

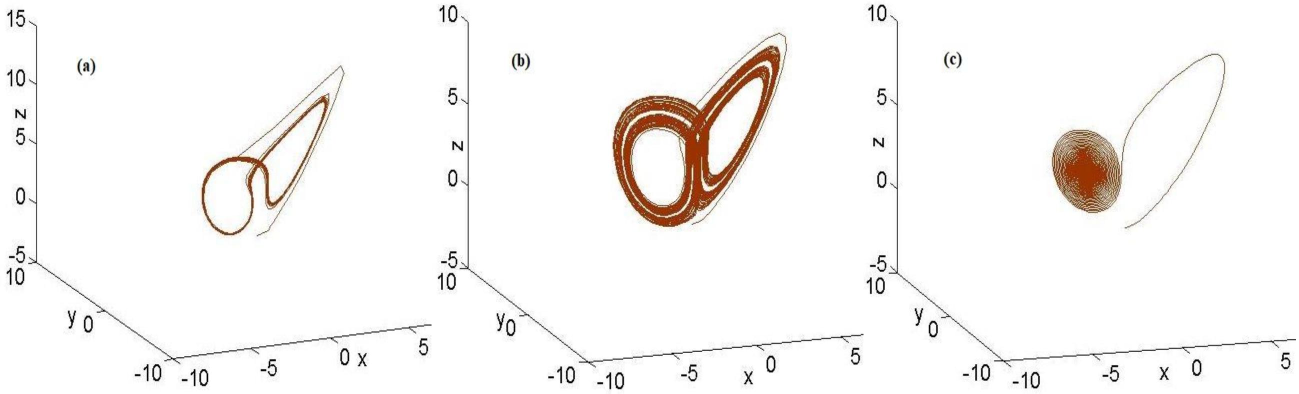

Figure 4: *Various Phase orbit diagrams of the System (3) with (a)* $\alpha = 0.7$ *, (b)* $\alpha = 0.8$*, (c)* $\alpha = 0.95$.

In recent years, one of the most important issue for researchers is dynamical analysis of models in engineering and biological. The existence of chaotic attractors is one of the most interesting dynamical phenomena. Poincare was the first one to notice that the existence of a chaotic attractor. In dynamical systems, an invariant subset of numerical values towards which a system moves to evolve for a varity of initial conditions of the system. The attractor vales remain close when the the system values get nearer to the attractor values even if the system is distrubed. Set of points, manifolds, fixed points or limit cylcles are some examples of simple attractors. Moreover, chaotic or strange or itinerant attractor are the interesting attractors, in which a dynamical system can roam around without repeating itself.

If a system has fractal structure, then the attractor is called strange. Strange attractors demonstrate self similarity. A strange attractor exhibits chaotic behavior when the system is sensitive to initial points. Thus a dydnamical system with a chaotic attractor is locally unstability and globally stability. Here we present some common chaotic attractors.

A plot of the numerical values, $\alpha = 0.99, r = 0.001, a = 30, b = 3, c = 20$ calculated from the discrete fractional order Chen system (3) and using particular initial values $(x_0 = -0.1; y_0 = 0.5; z_0 = -0.6)$. The orbit starting from any initial point approaches the paths shown in Figure-5, but the actual path depends only on the initial points of the attractor, and after any number of iterations the points lead to arbitrarily close together. Figure-6 assumes another set of values, $\alpha = 0.99, r = 0.001, a = 35, b = 3, c = 20$ with initial conditions $(x_0 = -0.1; y_0 = 0.5; z_0 = -0.6)$. Figure-6 displays an attractor.

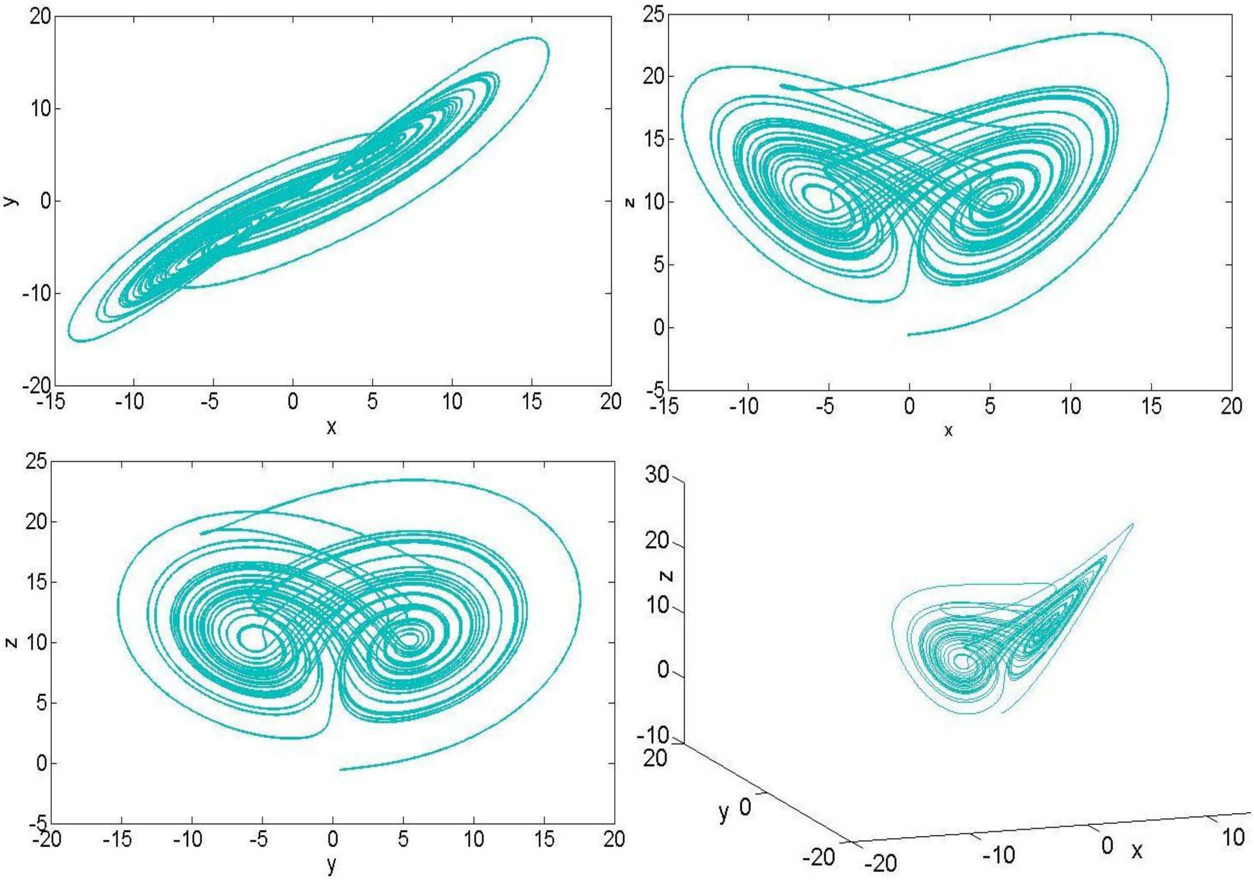

Figure 5: *Chaotic attractor of the System (3) with the parameter values* $\alpha = 0.99, r = 0.001, a = 30, b = 3, c = 20$ *and the initial conditions* $x = -0.1, y = 0.5, z = -0.6$

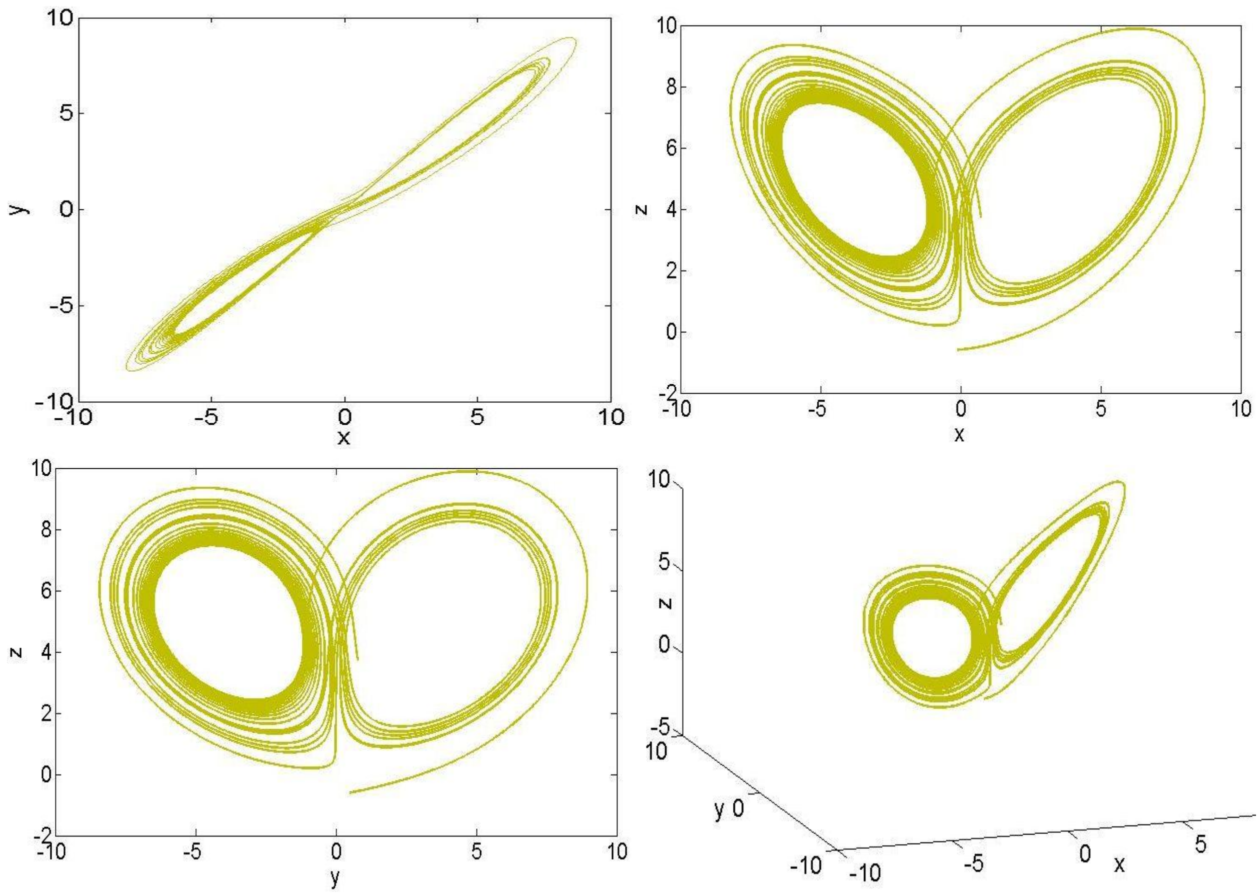

Figure 6: *Chaotic attractor of the System (3) with the numerical values* $\alpha = 0.99, r = 0.001, a = 35, b = 3, c = 20$ *and the initial points* $x = -0.1, y = 0.5, z = -0.6$

5. CONCLUSION

The purpose of this work is to investigate the dynamics and establish the existence of chaotic attractor in the discrete FO Chen system with order in (0,1). Moreover, we presented the description of the discrete FO Chen system which is developed from (1) by using a discretization method. We found three fixed points and Routh-Hurwitz method is applied to analyse the stability of the system (3). Numerically, we have shown the dynamical behaviour of the system (3) by plotting time lines and phase planes diagrams. Also we established the presence of chaotic attractors for different parameter values of discrete FO Chen System.

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