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Photo – gravitational Effect on the Sitnikov Five – body Problem forming Square Configuration

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Abstract— ***This paper deals with an extension of Sitnikov problem. Presently by considering the four radiating primaries at the vertices of a square and moving on the common circular orbit and the fifth body (infinitesimal mass) moving along the vertical line through the centre of the circular orbit, we form a new problem. We have developed the series solution of the Sitnikov five – body problem by the method of Lindstedt – Poincare. Also, we have examined the stability of the libration points with the help of the nature of roots of the characteristic equation. Thus, it is found that photo – gravitation has no effect on the Lindstedt – Poincare series solution but the libration points still stable.***

Keywords— Sitnikov Problem, Square Configuration, Lindstedt – Poincare Method, Series Solution, Stability

I. INTRODUCTION

The Sitnikov problem is a particular case of the restricted three-body problem in which two primaries with equal masses $(m_1 = m_2)$ move in a circular or an elliptic orbit around their common center of mass under the Newtonian force of attraction and the infinitesimal mass m (the infinitesimal mass is much less than the mass of the other two primaries) moves along the line perpendicular to the plane of motion of the primaries and passes through the center of mass of the primaries.

Pavanini (1907) introduced the problem for the first time as the special case of the circular Restricted three – body problem (CR3BP) and MacMillan (1913) expressed its solution in terms of Jacobi elliptic functions. After a long gap of almost half century Sitnikov (1960) studied the problem in detail and proved the existence of oscillating motion of the restricted three – body problem. Stumpff (1965) rediscussed the above problem. Sitnikov's problem has further been studied by many authors. Perdios et al. (1988) have studied stability and bifurcation of Sitnikov motion. Liu and Sun (1990) have studied the Sitnikov problem without taking the original differential equation and discovered an invariant set of hyperbolic solutions. Hagel (1992) has studied the problem by a new analytic approach. Faruque (2003) has established the new analytical expression for the position of the infinitesimal body in the elliptic Sitnikov problem.

Further by some author's, chaotic motion also have been studied. Perdios (2007) has studied the manifolds of families of three – dimensional periodic orbits in the three – body problem. Suraj and Hassan (2010) have averaged the equation of motion of the Sitnikov restricted four – body problem under the gravitational forces and they further extended the problem when all the primaries are sources of radiation. Shahbaz and Hassan (2014) have studied the connection between three – body configuration and four – body configuration of the Sitnikov problem when one of the masses approaches zero: Circular case. Further, Shahbaz and Hassan (2014) have studied Sitnikov cyclic configuration of $(n+1)$ body problem. Shahbaz, Bhatnagar and Hassan (2014) have studied Sitnikov problem cyclic kite configuration. Rahman, Garain and Hassan (2014) have studied solution and stability of restricted three – body problem, when the primaries are sources of radiation. Rahman, Garain and Hassan (2015) studied effect of oblateness of the primaries on the Sitnikov three – body problem.

At present, we proposed to study the effect of photo – gravitation on the motion of infinitesimal mass in the Sitnikov five – body problem when the primaries form a square configuration. Stability of libration points and Poincare section for periodicity has also been examined.

II. EQUATION OF MOTION

Let P_1, P_2, P_3 and P_4 be the four primaries of equal masses $\left(m_1 = m_2 = m_3 = m_4 = \frac{1}{4}\right)$ forming a square configuration $P_1P_2P_3P_4$. Let $P_1P_2 = P_2P_3 = P_3P_4 = P_4P_1 = l$. Since the masses of the primaries are equal hence we may assume that their centre of mass O to be at rest and consequently it is assumed as the origin and the primaries will move on common circular orbit with radius $OP_1 = OP_2 = OP_3 = OP_4 = a$ and common centre at O . Considering P_1OP_2 as the $x-$axis, P_2OP_4 as the $y-$axis and along the motion the infinitesimal mass as the $z-$axis. In such a system, the motion of the infinitesimal mass is one dimensional. Let at any time $t, P(0,0,z)$ be the position of the infinitesimal mass m and ω be the angular velocity of the frame about the origin O , then

$$\omega^2 l^3 = G\left(m_1+m_2+m_3+m_4\right)=G, \qquad \left[\text{as } m_1+m_2+m_3+m_4=1\right] \tag{1}$$

where G is the gravitational constant.

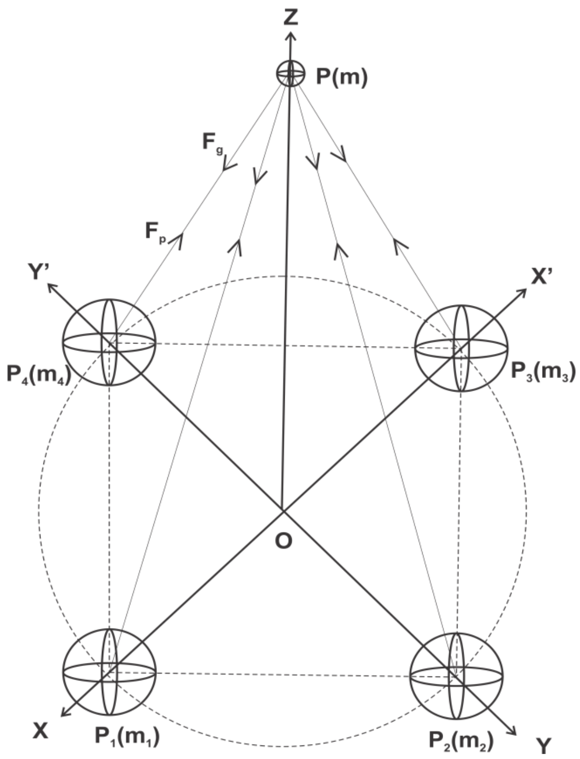

Fig. 1 Configuration of Sitnikov Five – body Problem

Let q_1, q_2, q_3 and q_4 be the radiation factors of the four primaries respectively, then the equation of motion of the infinitesimal mass in the photo – gravitational field of the four primaries P_1, P_2, P_3 and P_4 can be written as

$$\frac{d^2z}{dt^2} = -\frac{Gm_1zq_1}{r^3} - \frac{Gm_2zq_2}{r^3} - \frac{Gm_3zq_3}{r^3} - \frac{Gm_4zq_4}{r^3}, \tag{2}$$

where $PP_1 = PP_2 = PP_3 = PP_4 = r_i = \sqrt{z^2+a^2} = r.$

Since $0 < q_i << 1,\ i = 1,2,3,4,$ i.e., $1-q_i = p_i = 1-\frac{F_p}{F_g},$ hence the equation of motion (2) takes the form

$$\frac{d^2z}{dt^2} = -\frac{Gz\left(1-p\right)}{r^3}, \tag{3}$$

where $p = \frac{1}{4}\sum_{i=1}^{4} p_i,\ \sum_{i=1}^{4} m_i = 1,\ F_p$ is the radiating repulsion and F_g is the gravitational attraction of each primary.

Let us fix the unit of time and length in such a way that $G = 1$ and $l = 1,$ then equation (3) reduces to

$$\frac{d^2z}{dt^2} + \frac{\left(1-p\right)z}{\left(z^2+\frac{1}{2}\right)^{\frac{3}{2}}} = 0, \tag{4}$$

where $2a^2 = l^2 = 1,$ So $a^2 = \frac{1}{2}$ and from Equation (1) $\omega^2 = 1.$

By using binomial theorem,

$$\frac{d^2z}{dt^2} + 2\sqrt{2}\left(1-p\right)z\left(1-3z^2+...\right) = 0\,.$$

As $z << 1,$ so neglecting higher order terms of z above the third and hence the above equation takes the form

$$\frac{d^2z}{dt^2} + 2\sqrt{2}\left(1-p\right)z - 6\sqrt{2}\left(1-p\right)z^3 = 0. \tag{5}$$

III. Series Solution By Lindstedt – Poincare Method

To find the solution for z as a function of time t, let us solve the Equation (5) whose exact analytical solution is not possible and thus we shall try to find the approximate solution. For the first approximation, we assume that the energy of the infinitesimal mass is such that it is bound to remain near the origin. The linearized form of Equation (5) is

$$\frac{d^2z}{dt^2}+\eta_0^2 z-\varepsilon z^3=0, \tag{6}$$

where $\eta_0^2=2\sqrt{2}\left(1-p\right),\varepsilon=6\sqrt{2}\left(1-p\right)$.

To follow Lindstedt – Poincare method, let us write

$$z=\sum_{r=0}^{\infty}\varepsilon^r z_r. \tag{7}$$

Now we introduce an independent variable τ defined by $\tau=\eta t$ where

$$\eta=\sum_{r=0}^{\infty}\varepsilon^r \eta_r. \tag{8}$$

Using $\tau=\eta t$ in the Equation (6), we get

$$\eta^2\frac{d^2z}{d\tau^2}+\eta_0^2 z-\varepsilon z^3=0. \tag{9}$$

Using Equations (7) and (8) in Equation (9) and equating the coefficient of like powers of ε, we get the differential equations as follows

$$\eta_0^2\left(\frac{d^2z_0}{d\tau^2}+z_0\right)=0, \tag{10}$$

$$\eta_0^2\left(\frac{d^2z_1}{d\tau^2}+z_1\right)+2\eta_0\eta_1\frac{d^2z_0}{d\tau^2}-z_0^3=0, \tag{11}$$

$$\eta_0^2\left(\frac{d^2z_2}{d\tau^2}+z_2\right)+2\eta_0\eta_1\frac{d^2z_1}{d\tau^2}+\left(\eta_1^2+2\eta_0\eta_2\right)\frac{d^2z_0}{d\tau^2}-3z_0^2z_1=0, \tag{12}$$

$$\eta_0^2\left(\frac{d^2z_3}{d\tau^2}+z_3\right)+2\eta_0\eta_1\frac{d^2z_2}{d\tau^2}+\left(\eta_1^2+2\eta_0\eta_2\right)\frac{d^2z_1}{d\tau^2}+2\left(\eta_1\eta_2+\eta_0\eta_3\right)\frac{d^2z_0}{d\tau^2}-3\left(z_0^2z_2+z_0z_1^2\right)=0. \tag{13}$$

The general solution of Equation (10) is

$z_0=K_1\cos\tau+K_2\sin\tau,$

where K_1 and K_2 are arbitrary constants of integration.

Using the initial conditions $z(0)=K$ and $\dot{z}(0)=0,$ the complete solution of Equation (10) can be written as

$z_0=K\cos\tau,$ the first approximate value of z. (14)

Substituting the values of z_0 and $\ddot{z}_0$ in Equation (11), we get

$$\frac{d^2z_1}{d\tau^2}+z_1=\left(\frac{3K^3}{4\eta_0^2}+\frac{2K\eta_1}{\eta_0}\right)\cos\tau+\frac{K^3}{4\eta_0^2}\cos3\tau. \tag{15}$$

To avoid the secular term, equating to zero the coefficient of $\cos\tau,$ we get

$\eta_1=-\dfrac{3K^2}{8\eta_0}$ and hence Equation (15) reduced to

$$\frac{d^2z_1}{d\tau^2}+z_1=\frac{K^3}{4\eta_0^2}\cos3\tau. \tag{16}$$

The general solution of the Equation (15) is

$$z_1=K_3\cos\tau+K_4\sin\tau-\frac{K^3}{32\eta_0^2}\cos3\tau, \tag{17}$$

where K_3 and K_4 are arbitrary constants of integration.

Thus the second approximate value of z is given by

$z=z_0+\varepsilon z_1,$

$$z=K\cos\tau+\varepsilon K_3\cos\tau+\varepsilon K_4\sin\tau-\frac{\varepsilon K^3}{32\eta_0^2}\cos3\tau. \tag{18}$$

Using the initial conditions $z(0)=K$ and $\dot{z}(0)=0$ in Equation (18), we get

$$z = K\cos\tau + \frac{\varepsilon K^3}{32\eta_0^2}\cos\tau - \frac{\varepsilon K^3}{32\eta_0^2}\cos 3\tau. \tag{19}$$

With the help of Equations (18) and (19), one can find

$$z_1 = \frac{K^3}{32\eta_0^2}\left(\cos\tau - \cos 3\tau\right).$$

Substituting the values of $z_0, \ddot{z}_0, z_1, \ddot{z}_1, \eta_1$ in Equation (12), we get

$$\frac{d^2 z_2}{d\tau^2} + z_2 = \left(\frac{21K^5}{128\eta_0^4} + \frac{2K\eta_2}{\eta_0}\right)\cos\tau + \frac{3K^5}{16\eta_0^4}\cos 3\tau - \frac{3K^5}{128\eta_0^4}\cos 5\tau.$$

To avoid the secular term, equating to zero the coefficient of $\cos\tau$, we get $\eta_2 = -\frac{21K^4}{256\eta_0^3}$ and thus

$$\frac{d^2 z_2}{d\tau^2} + z_2 = \frac{3K^5}{16\eta_0^4}\cos 3\tau - \frac{3K^5}{128\eta_0^4}\cos 5\tau. \tag{20}$$

The general solution of the Equation (20) is

$$z_2 = K_5\cos\tau + K_6\sin\tau - \frac{3K^5}{128\eta_0^4}\cos 3\tau + \frac{K^5}{1024\eta_0^4}\cos 5\tau, \tag{21}$$

where K_5 and K_6 are arbitrary constants of integration.

The third approximate value of z is given by

$$z = z_0 + \varepsilon z_1 + \varepsilon^2 z_2,$$

$$z = K\cos\tau + \frac{\varepsilon K^3}{32\eta_0^2}\left(\cos\tau - \cos 3\tau\right) + \varepsilon^2\left(K_5\cos\tau + K_6\sin\tau - \frac{3K^5}{128\eta_0^4}\cos 3\tau + \frac{K^5}{1024\eta_0^4}\cos 5\tau\right). \tag{22}$$

Using the initial conditions $z(0) = K$ and $\dot{z}(0) = 0$ in Equation (22), we get

$$z = K\cos\tau + \frac{\varepsilon K^3}{32\eta_0^2}\left(\cos\tau - \cos 3\tau\right) + \varepsilon^2\left(\frac{23K^5}{1024\eta_0^4}\cos\tau - \frac{3K^5}{128\eta_0^4}\cos 3\tau + \frac{K^5}{1024\eta_0^4}\cos 5\tau\right). \tag{23}$$

With the help of Equations (22) and (23), one can find

$$z_2 = \frac{K^5}{1024\eta_0^4}\left(23\cos\tau - 24\cos 3\tau + \cos 5\tau\right). \tag{24}$$

Substituting the values of $z_0, \ddot{z}_0, z_1, \ddot{z}_1, z_2, \ddot{z}_2, \eta_1, \eta_2$ in Equation (13), we get

$$\frac{d^2 z_3}{d\tau^2} + z_3 = \left(\frac{81K^7}{1024\eta_0^6} + \frac{2K\eta_3}{\eta_0}\right)\cos\tau + \frac{297K^7}{2048\eta_0^6}\cos 3\tau - \frac{9K^7}{256\eta_0^6}\cos 5\tau + \frac{3K^7}{2048\eta_0^6}\cos 7\tau.$$

To avoid the secular term, we shall put the coefficient of $\cos\tau$ equal to zero, thus

$\eta_3 = -\frac{81K^6}{2048\eta_0^5}$ and

$$\frac{d^2 z_3}{d\tau^2} + z_3 = \frac{297K^7}{2048\eta_0^6}\cos 3\tau - \frac{9K^7}{256\eta_0^6}\cos 5\tau + \frac{3K^7}{2048\eta_0^6}\cos 7\tau. \tag{25}$$

The general solution of the Equation (25) is

$$z_3 = K_7\cos\tau + K_8\sin\tau - \frac{3K^7}{98304\eta_0^6}\left(594\cos 3\tau - 48\cos 5\tau + \cos 7\tau\right).$$

Thus the fourth approximate value of z is given by

$$z = z_0 + \varepsilon z_1 + \varepsilon^2 z_2 + \varepsilon^3 z_3,$$

$$\begin{aligned} z &= K\cos\tau + \frac{\varepsilon K^3}{32\eta_0^2}\left(\cos\tau - \cos 3\tau\right) + \frac{\varepsilon^2 K^5}{1024\eta_0^4}\left(23\cos\tau - 24\cos 3\tau + \cos 5\tau\right) \\ &\quad + \varepsilon^3\left\{K_7\cos\tau + K_8\sin\tau - \frac{3K^7}{98304\eta_0^6}\left(594\cos 3\tau - 48\cos 5\tau + \cos 7\tau\right)\right\}. \end{aligned} \tag{26}$$

Using the initial conditions $z(0) = K$ and $\dot{z}(0) = 0$ in Equation (26), we get

$$\left.\begin{aligned} z &= K\cos\tau + \frac{\varepsilon K^3}{32\eta_0^2}\left(\cos\tau - \cos 3\tau\right) + \frac{\varepsilon^2 K^5}{1024\eta_0^4}\left(23\cos\tau - 24\cos 3\tau + \cos 5\tau\right) \\ &+ \frac{\varepsilon^3 K^7}{3276\eta_0^6}\left(547\cos\tau - 594\cos 3\tau + 48\cos 5\tau - \cos 7\tau\right). \end{aligned}\right\} \tag{27}$$

With the help of Equations (26) and (27), one can find

$$z_3 = \frac{K^7}{32768\eta_0^6}\left(547\cos\tau - 594\cos 3\tau + 48\cos 5\tau - \cos 7\tau\right). \tag{28}$$

Proceeding in this way, we can obtain the remaining consecutive terms and the value of z can be written as

$$z = z_0 + \varepsilon z_1 + \varepsilon^2 z_2 + \varepsilon^3 z_3 + ...,$$

$$\begin{aligned} z &= \left(\cos\eta t\right)K + \left(\frac{1}{288}\right)\left(\cos\eta t - \cos 3\eta t\right)K^3 + \left(\frac{1}{9216}\right)\left(23\cos\eta t - 24\cos 3\eta t + \cos 5\eta t\right)K^5 \\ &+ \left(\frac{1}{884736}\right)\left(547\cos\eta t - 594\cos 3\eta t + 48\cos 5\eta t - \cos 7\eta t\right)K^7 + \ldots \end{aligned} \tag{29}$$

Thus the series solution of Equation (29) shows no effect of photo – gravitation.

IV. Stability Of The Equilibrium Points

Following Murray and Dermott (1999), let us check the Stability of the Sitnikov motion. We rewrite the general equations of motion given in Equation (5) as

$$\left.\begin{aligned} \ddot{x} - 2n\dot{y} &= \Omega_x, \\ \ddot{y} + 2n\dot{x} &= \Omega_y, \\ \ddot{z} &= \Omega_z, \end{aligned}\right\} \tag{30}$$

where the force function Ω is given by

$$\Omega = -\left(1-p\right)\left(z^2 + \frac{1}{2}\right)^{-\frac{1}{2}},\ \Omega_z = -\left(1-p\right)z\left(z^2 + \frac{1}{2}\right)^{-\frac{3}{2}} \tag{31}$$

and

$$\Omega_{zz} = -2\sqrt{2}\left(1-p\right) + 18\sqrt{2}\left(1-p\right)z^2 - 60\left(1-p\right)z^4. \tag{32}$$

The system of Equation (30) can be written as

$$\left.\begin{aligned} \ddot{x} - 2n\dot{y} &= \Omega_x = f\left(x, y, z\right) \text{ say}, \\ \ddot{y} - 2n\dot{x} &= \Omega_y = g\left(x, y, z\right) \text{ say}, \\ \ddot{z} &= \Omega_z = h\left(x, y, z\right) \text{ say} \end{aligned}\right\} \tag{33}$$

In stationary solution, Ω is a function of z only, so there is no solution in the xy – plane, clearly the solution lie on the z – axis only. Let us denote the libration point as $P\left(0, 0, z_0\right)$ then from Equation (33), we have

$$\left.\begin{aligned} \Omega_x^0 &= f\left(0, 0, z_0\right) = 0, \\ \Omega_y^0 &= g\left(0, 0, z_0\right) = 0, \\ \Omega_z^0 &= h\left(0, 0, z_0\right) = 0 = -\left(1-p\right)z\left(z_0^2 + \frac{1}{2}\right)^{-\frac{3}{2}}, \end{aligned}\right\} \tag{34}$$

where $\Omega_x^0, \Omega_y^0, \Omega_z^0$, are the values of $\Omega_x, \Omega_y, \Omega_z$ at the libration points.

We shall now communicate the small displacement ξ, η, ζ in the coordinate of P such that

$$x = 0 + \xi,\ y = 0 + \eta,\ z = z_0 + \zeta.$$

Equation (33) becomes

$$\begin{aligned} \ddot{\xi} - 2n\dot{\eta} &= f\left(0+\xi, 0+\eta, z_0+\zeta\right), \\ \ddot{\eta} + 2n\xi &= g\left(0+\xi, 0+\eta, z_0+\zeta\right), \\ \ddot{\zeta} &= h\left(0+\xi, 0+\eta, z_0+\zeta\right). \end{aligned}$$

Now applying the Taylor's theorem in the neighbourhood of $\left(0, 0, z_0\right)$, we get

$$\left.\begin{aligned}\ddot{\xi}-2n\dot{\eta}&=f\left(0,0,z_0\right)+\xi\left(\frac{\partial f}{\partial x}\right)_0+\eta\left(\frac{\partial f}{\partial y}\right)_0+\zeta\left(\frac{\partial f}{\partial z}\right)_0+\text{higher order infinitesimals},\\ \ddot{\eta}+2n\dot{\xi}&=g\left(0,0,z_0\right)+\xi\left(\frac{\partial g}{\partial x}\right)_0+\eta\left(\frac{\partial g}{\partial y}\right)_0+\zeta\left(\frac{\partial g}{\partial z}\right)_0+\text{higher order infinitesimals},\\ \ddot{\zeta}&=h\left(0,0,z_0\right)+\xi\left(\frac{\partial h}{\partial x}\right)_0+\eta\left(\frac{\partial h}{\partial y}\right)_0+\zeta\left(\frac{\partial h}{\partial z}\right)_0+\text{higher order infinitesimals}.\end{aligned}\right\}\tag{35}$$

By using Equation (5), the system of Equation (35) reduced to

$$\begin{aligned}\ddot{\xi}-2n\dot{\eta}&=\xi\frac{\partial}{\partial x}\left(\frac{\partial\Omega}{\partial x}\right)_0+\eta\frac{\partial}{\partial y}\left(\frac{\partial\Omega}{\partial x}\right)_0+\zeta\frac{\partial}{\partial z}\left(\frac{\partial\Omega}{\partial x}\right)_0+\text{higher order infinitesimals},\\ \ddot{\eta}+2n\dot{\xi}&=\xi\frac{\partial}{\partial x}\left(\frac{\partial\Omega}{\partial y}\right)_0+\eta\frac{\partial}{\partial y}\left(\frac{\partial\Omega}{\partial y}\right)_0+\zeta\frac{\partial}{\partial x}\left(\frac{\partial\Omega}{\partial y}\right)_0+\text{higher order infinitesimals},\\ \ddot{\zeta}&=\xi\frac{\partial}{\partial x}\left(\frac{\partial\Omega}{\partial z}\right)_0+\eta\frac{\partial}{\partial y}\left(\frac{\partial\Omega}{\partial z}\right)_0+\zeta\frac{\partial}{\partial z}\left(\frac{\partial\Omega}{\partial z}\right)_0+\text{higher order infinitesimals}.\end{aligned}$$

where $f=\frac{\partial\Omega}{\partial x}$, $g=\frac{\partial\Omega}{\partial y}$ and $h=\frac{\partial\Omega}{\partial z}$.

Neglecting the higher order terms of ξ,η and ζ, we get the new variational equations

$$\left.\begin{aligned}\ddot{\xi}-2n\dot{\eta}&=\xi\Omega_{xx}^0+\eta\Omega_{yx}^0+\zeta\Omega_{zx}^0,\\ \ddot{\eta}+2n\dot{\xi}&=\xi\Omega_{xy}^0+\eta\Omega_{yy}^0+\zeta\Omega_{zy}^0,\\ \ddot{\zeta}&=\xi\Omega_{xz}^0+\eta\Omega_{yz}^0+\zeta\Omega_{zz}^0,\end{aligned}\right\}\tag{36}$$

where $\Omega_{xx}^0,\Omega_{xy}^0,\Omega_{xz}^0,\ldots$ represent the second order derivatives of Ω at the libration points.

The system of Equation (36) can be written in the form of a single matrix equation as

$$\dot{X}=AX,\tag{37}$$

where

$$A=\begin{bmatrix}0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\\Omega_{xx}^0&\Omega_{yx}^0&\Omega_{zx}^0&0&2n&0\\\Omega_{xy}^0&\Omega_{yy}^0&\Omega_{zy}^0&-2n&0&0\\\Omega_{xz}^0&\Omega_{yz}^0&\Omega_{zz}^0&0&0&0\end{bmatrix}\text{ and }X=\begin{bmatrix}\xi\\\eta\\\zeta\\\dot{\xi}\\\dot{\eta}\\\dot{\zeta}\end{bmatrix}.$$

If any matrix X satisfy the equation $AX=\lambda X$, (38)

then X is said to be an Eigen vector of the matrix A and scalar λ is its corresponding Eigen value. If A is thought of as a transformation matrix, then the result of applying A to the particular vector X satisfying Equation (38) is to produce a vector in the same direction as X, but of a different magnitude.

Now the Equation (38) can be written as $\left(A-\lambda I\right)X=0.$ The set of six simultaneous linear equations in six unknowns $\xi,\eta,\zeta,\dot{\xi},\dot{\eta},\dot{\zeta}$ will have non trivial solutions provided the determinant of the matrix $\left(A-\lambda I\right)$ vanishes.

$$\text{i.e.,}\left|A-\lambda I\right|=0.\tag{39}$$

Now the non – trivial solution of Equation (39) will be stable if they are periodic along with their solutions and if the Eigen values of the matrix A are either zero or imaginary. The Equation (39) yields

$$\lambda^2\left(\lambda^2+4n^2\right)\left(\lambda^2-\Omega_{zz}^0\right)=0.\tag{40}$$

The Equation (40) is a polynomial equation of degree six in λ so there will be three roots in λ^2 corresponding to the three factors of Equation (40). The conditions for stable solutions are

$\lambda_i^2\le 0\ \left(i=1,2,3\right),$

where $\lambda_1^2=0,\lambda_2^2=-4n^2,\lambda_3^2=\Omega_{zz}^0$ are three roots of Equation(40). Since $\lambda_1^0=0,$ hence $\lambda_{11}=\lambda_{12}=0.$

When $\lambda_2^2=-4n^2<0,$ then $\lambda_2=\pm 2ni$.

i.e., $\lambda_{21}=2ni$ and $\lambda_{22}=\pm 2ni$.

When $\lambda_3^2=\Omega_{zz}^0$. Since $z_0<<1$ hence the quantity containing higher power of z_0 above the second must be neglected, therefore

$\lambda_3^2 = \Omega_{zz}^0 = -2\sqrt{2}(1-p)(1+9z_0^2) < 0,$

$\lambda_{31} = \mathrm{i}\left\{2\sqrt{2}(1-p)(1+9z_0^2)\right\}^{\frac{1}{2}}$ and $\lambda_{32} = -\mathrm{i}\left\{2\sqrt{2}(1-p)(1+9z_0^2)\right\}^{\frac{1}{2}}.$ (imaginary)

Thus all the six roots of the characteristic equations are

$0,\ 0,\ 2ni,\ -2ni,\ i\left\{2\sqrt{2}(1-p)(1+9z_0^2)\right\}^{\frac{1}{2}},\ -\left\{2\sqrt{2}(1-p)(1+9z_0^2)\right\}^{\frac{1}{2}}.$

i.e., they are either zero or imaginary and hence the equilibrium positions of the Sitnikov restricted three – body problem are stable.

V. Discussion And Conclusions

About the theoretical evolution given in section 1, we have derived the non – linear equations of motion of the infinitesimal mass in the photo – gravitational field of the four radiating primaries of equal masses situated at the vertices of a square and moving on a common circular orbit of radius a in section 2. The infinitesimal mass at $P(0,0,z)$ equidistant from the four primaries which is given by $r_i = PP_i = \sqrt{z^2 + a^2} = r\,(\text{say})\ i = 1,2,3,4$. In section 3, we have linearized the non – linear equation of motion to get the standard form given by Lindstedt – Poincare in order to get the series solution. The series solution by Lindstedt – Poincare method is given in Equation (29). To examine the stability of the five collinear libration points lying on the z – axis, we solved the characteristic equation $\lambda^2(\lambda^2 + 4n^2)(\lambda^2 - \Omega_{zz}^0) = 0$. The nature of roots of the characteristic equation decides the stability of the libration points. In our case, all the five collinear libration points are stable.

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