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Discrete Fractional Order SIR Epidemic Model of Childhood Diseases with Constant Vaccination and it's Stability

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Abstract— ***In this paper, we propose to study an epidemic model of childhood disease in a human population. Fractional order SIR epidemic model is considered and its discrete form is obtained. Local asymptotic stability of disease free equilibrium and endemic equilibrium points are discussed and the basic reproduction number $\Re_0$ is obtained via next generation matrix method. Time plots and phase portraits are analyzed under suitable conditions. Numerical examples are used to verify the stability results.***

Keywords — fractional order, SIR model, stability, discretization process

I. Introduction

Mathematical models have become an indispensable tool in analyzing the spread and control of infectious diseases. Mathematical models in epidemiology attempts to model, study and analyze the disease propagation in a population. In 1927, William Kermack and Anderson McKendrick introduced compartmental models and published in "Contribution to the mathematical theory of epidemics". They introduced the SIR model, the total population $N(t)$ is partitioned into three compartments, S -Susceptible, I -Infected and R -Recovered [1,3].

II. FRACTIONAL CALCULUS

Fractional calculus is the field of mathematical analysis which deals with the theory and applications of integrals and derivatives of arbitrary order. One of the fundamental problems in control is the stability analysis of the dynamic system. The stability problem for linear, continuous time, discrete time fractional order systems has stimulated the interest of mathematicians and researchers, as its applications vary from numerical analysis to applied fields of engineering, science, economics and finance [2, 4, 8]. The dynamics of the SIR model is described in the flow diagram as follows:

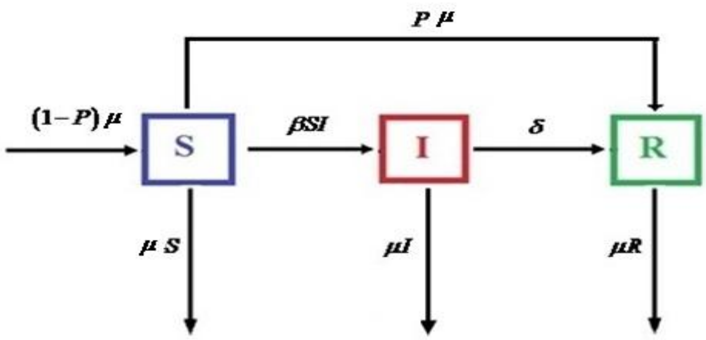

Figure 1: Flow diagram of SIR epidemic model

In this paper, we consider the fractional order differeintional equations [5, 6] of the form

$$\begin{aligned} D^{\alpha}S(t) &= (1-P)\mu - \beta S(t)I(t) - \mu S(t), \\ D^{\alpha}I(t) &= \beta S(t)I(t) - (\delta+\mu)I(t), \\ D^{\alpha}R(t) &= P\mu + \delta I(t) - \mu R(t) \end{aligned} \tag{1}$$

where μ is the natural birth and death rates, P (with $0 < P < 1$) is the fraction of citizens vaccinated at birth each year, β is the contact rate, δ is the recovery rate, α is the fractional order $\alpha \in (0,1]$, $(S(0) = S_0, I(0) = I_0, R(0) = R_0)$ are the initial values, h is the step size. This fractional order SIR model considers the efficacy of the vaccine is 100%.
Applying the discretization process for a fractional order system described in [7, 9], we obtain the discrete fractional order SIR epidemic model as follows:

$$\begin{aligned} S_{n+1} &= S_n + \frac{h^\alpha}{\Gamma(1+\alpha)}\left((1-P)\mu - \beta S_n I_n - \mu S_n\right), \\ I_{n+1} &= I_n + \frac{h^\alpha}{\Gamma(1+\alpha)}\left(\beta S_n I_n - (\delta+\mu) I_n\right), \\ R_{n+1} &= R_n + \frac{h^\alpha}{\Gamma(1+\alpha)}\left(P\mu + \delta I_n - \mu R_n\right). \end{aligned} \tag{2}$$

III. EQUILIBRIUM POINTS AND BASIC REPRODUCTIVE NUMBER

Our model (2) has two equilibrium points:

(i) Disease Free Equilibrium (DFE) point $E_0 = \left(1-P, 0, P\right)$

(ii) Endemic Equilibrium (EE) point $E_1 = \left(S^*, I^*, R^*\right)$,

where $S^* = \dfrac{\delta+\mu}{\beta}$, $I^* = \dfrac{(1-P)\mu}{\delta+\mu} - \dfrac{\mu}{\beta}$, $R^* = P + \dfrac{\delta(1-P)}{\delta+\mu} - \dfrac{\delta}{\beta}$.

Using the next generation matrix are obtain the basic reproduction number $\Re_0 = \dfrac{\beta(1-P)}{\delta+\mu}$.

IV. STABILITY ANALYSIS WITH SIMULATIONS

In this section, we shall study the equilibrium solutions of model (2) and their local stability. We compute the Jacobian matrix corresponding to each equilibrium point and Jury conditions are used to analyze the stability.

Lemma 1. [Jury Conditions]Suppose the characteristic polynomial is $P(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$. The solutions $\lambda_i, i = 1,2,3$, of $P(\lambda) = 0$ satisfy $|\lambda_i| < 1$ iff the following three conditions hold:

(i) $P(1) = 1 + a_1 + a_2 + a_3 > 0$

(ii) $(-1)^3 P(-1) = 1 - a_1 + a_2 - a_3 < 0$

(iii) $1 - (a_3)^2 > |a_2 - a_3 a_1|$ (or) $a_3 < 1$

Lemma 2. Let $P(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$ where λ_1, λ_2 and λ_3 are the three roots. The topological properties of the equilibrium of the model (2) are

(i) $|\lambda_1| < 1$, $|\lambda_2| < 1$ and $|\lambda_3| < 1$ then $E\left(S^*, I^*, R^*\right)$ is a sink (locally asymptotic stable).

(ii) $|\lambda_1| > 1$, $|\lambda_2| > 1$ and $|\lambda_3| > 1$ then $E\left(S^*, I^*, R^*\right)$ is a source (unstable).

(iii) $|\lambda_1| > 1$, $|\lambda_2| < 1$ and $|\lambda_3| < 1$ (or $|\lambda_1| < 1, |\lambda_2| > 1$ and $|\lambda_3| > 1$) then $E\left(S^*, I^*, R^*\right)$ is a saddle (unstable).

(iv) $|\lambda_1| = 1$, $|\lambda_2| = 1$ and $|\lambda_3| = 1$ then $E\left(S^*, I^*, R^*\right)$ is non-hyperbolic.

Theorem 1. If $0 < \Re_0 < 1$, the DFE point E_0 of the model (2) is Locally Asymptotically Stable(LAS) and the DFE point E_0 is unstable if $\Re_0 > 0$.

Proof. The characteristic equation of the model (2) about DFE point E_0 is

$$\begin{aligned} P(\lambda) &= \lambda^3 + \lambda^2[H(\omega+2\mu) - 3] \\ &+ \lambda[3 - 2H(\omega+2\mu) + H^2\mu(2\omega+\mu)] \\ &+ [H^3\omega\mu^2 - H^2(2\omega+\mu)\mu + H(2\mu+\omega) - 1] \end{aligned}$$

where $H = \dfrac{h^\alpha}{\Gamma(1+\alpha)}$, $\omega = P\beta + \delta + \mu - \beta$. Let

$$\begin{aligned} a_1 &= H(\omega+2\mu) - 3, \\ a_2 &= 3 - 2H(\omega+2\mu) + H^2\mu(2\omega+\mu), \\ a_3 &= H^3\omega\mu^2 - H^2(2\omega+\mu)\mu + H(2\mu+\omega) - 1. \end{aligned}$$

The eigen values evaluated at E_0 are

$$\lambda_{1,2} = 1 - H\mu, \lambda_3 = 1 - H\omega.$$

From the Jury conditions, if $P(1) > 0, P(-1) < 0$, $a_3 < 1$ and the roots of $P(\lambda)$ satisfy $|\lambda_j| < 1$ and $0 < \Re_0 < 1$ then DFE point E_0 is LAS. Suppose $P(1) < 0$ and $\Re_0 > 1$ then DFE point E_0 is unstable.

Proposition 1. The DFE point E_0 of the model (2) is the following conditions hold,

(i) E_0 is a sink if $0<\mu H<2$ and $0<H\omega<2$.

(ii) E_0 is a source if $\mu H>2$ and $H\omega>2$.

(iii) E_0 is a saddle if $0<\mu H<2$ and $H\omega>2$ (or) $H\mu>2$ and $0<H\omega<2$.

(iv) E_0 is a non-hyperbolic if $\mu H=2$ and $H\omega=2$.

We consider the initial values $S_0=0.94,\ I_0=0.06,\ R_0=0.0$ for numerical study.

Example 1. In case 1, since $P(1)=0.0000022428>0$, $P(-1)=-7.7792<0$, $a_3=-0.9452<1$ and $\Re_0=0.9545<1$ then the DFE point $E_0=(0.7,0,0.3)$ of the model (2) is LAS. (see Fig.2). **In case 2**, since $P(1)=0.00000033642>0$, $P(-1)=-7.9165<0$, $a_3=-0.9792<1$ and $\Re_0=0.6<1$, the DFE point $E_0=(0.1,0,0.9)$ of the model (2) is LAS (see Fig.3).

TABLE I. PARAMETERS FOR NUMERICAL SIMULATIONS OF THE MODEL (2).

Parameters	P	β	δ	μ	h	α	E_0
Case 1	0.3	0.75	0.35	0.2	0.1	0.9	$(0.7,0,0.3)$
Case 2	0.9	0.9	0.1	0.05	0.1	0.9	$(0.1,0,0.9)$

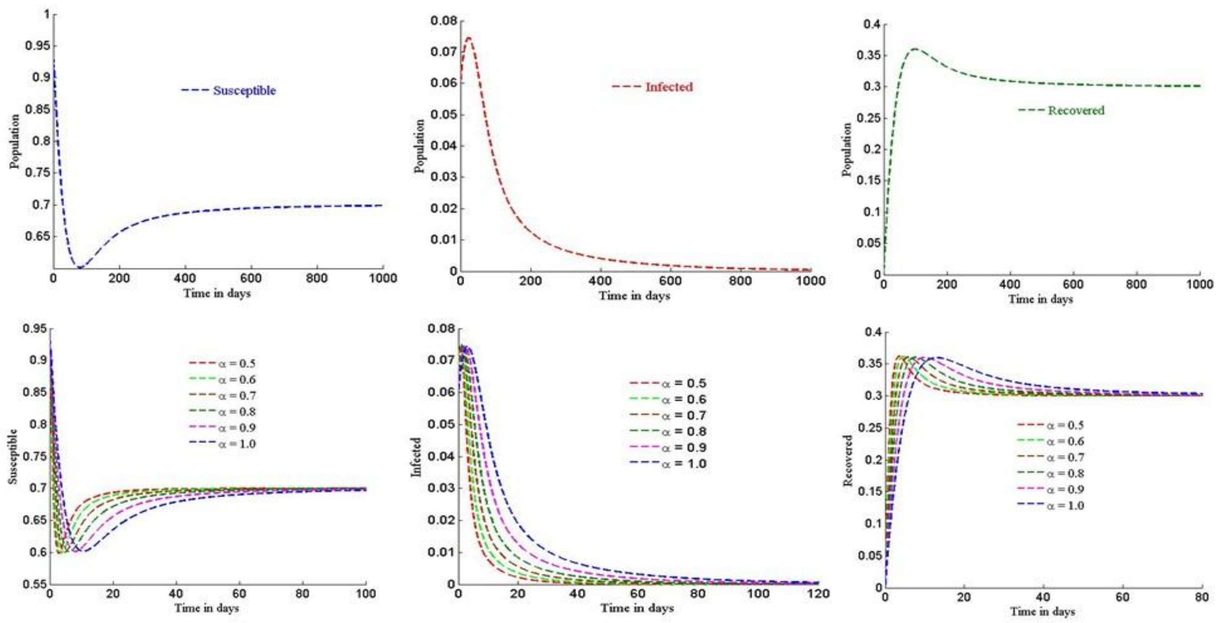

Fig. 2 Time plots of DFE point E_0 for different fractional order $\alpha \in (0.5,1.0]$ with stability $\Re_0<1$.

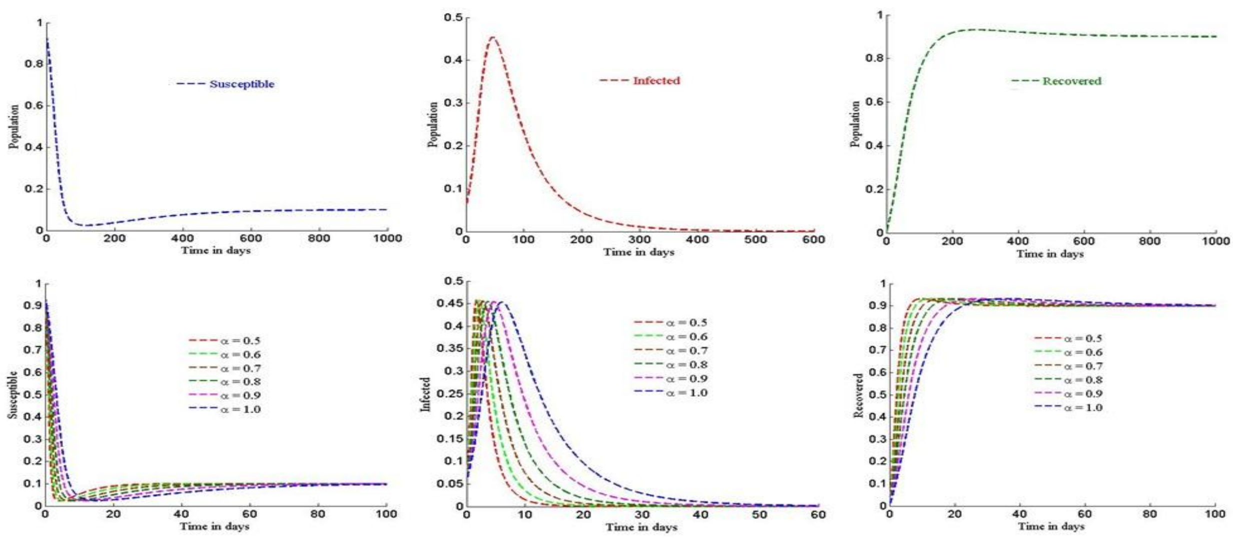

Fig. 3 Time plots of DFE point E_0 for different fractional order $\alpha \in (0.5,1.0]$ with stability $\Re_0<1$.

Theorem 2. The EE point E_1 of the model (2) is LAS if $\Re_0 > 1$.

Proof. The characteristic equation of the model (2) about EE point E_1 is

$$\begin{aligned} P(\lambda) &= \lambda^3 + \lambda^2[H(2\mu+\eta)-3] \\ &+\lambda[3-2H\mu-2H(\mu+\eta)+H^2\mu(\mu+\eta)+H^2(\delta+\mu)\eta] \\ &+[H(\eta+2\mu)-H^2\mu(\eta+\mu)-H^2(1-H\mu)(\delta+\mu)\eta-1]. \end{aligned}$$

where $\eta = \dfrac{\beta\mu(1-P)}{(\mu+\delta)} - \mu$. Let

$$\begin{aligned} a_{11} &= H(2\mu+\eta)-3, \\ a_{22} &= 3-2H\mu-2H(\mu+\eta)+H^2\mu(\mu+\eta)+H^2(\delta+\mu)\eta, \\ a_{33} &= H(\eta+2\mu)-H^2\mu(\eta+\mu)-H^2\eta(\delta+\mu)(1-H\mu)-1. \end{aligned}$$

The eigen values evaluated at E_1 are

$$\begin{aligned} \lambda_1 &= 1-H\mu, \\ \lambda_{2,3} &= \frac{2-H(\mu+\eta)}{2} \pm \frac{1}{2}\sqrt{(2-H(\mu+\eta))^2-4[H^2(\delta+\mu)\eta-H(\mu+\eta)+1]}. \end{aligned}$$

From the Jury conditions, if $P(1)>0, P(-1)<0$ and $a_{33}<1$, and the roots of $P(\lambda)$ satisfy $|\lambda_j|<1$ and $\Re_0>1$ then the EE point E_1 is LAS. Suppose $P(1)<0$ then EE point E_1 is unstable.

Proposition 2. The EE point E_1 of the model (2) is the following properties.

(i) E_1 is a sink if $0<\mu H<2$ and $0<H^2(\delta+\mu)\eta<2H(\eta+\mu)-4$.

(ii) E_1 is a source if $\mu H>2$ and $H^2(\delta+\mu)\eta>2H(\eta+\mu)-4$.

(iii) E_1 is a saddle if $0<\mu H<2$ and $H^2(\delta+\mu)\eta>2H(\eta+\mu)-4$

(or) $\mu H>2$ and $0<H^2(\delta+\mu)\eta<2H(\eta+\mu)-4$.

(iv) E_1 is a non-hyperbolic if $\mu H=2$ and $H^2(\delta+\mu)\eta=2H(\mu+\eta)-4$.

Example 2. In case 3, we have $P(1)=0.00000033642>0$, $P(-1)=-7.9598<0$, $a_{33}=-0.9900<1$ and $\Re_0=1.5833>1$ and the EE point $E_1=(0.6, 0.0219, 0.3781)$ of the model (2) is LAS (see Fig. 4 & 5).

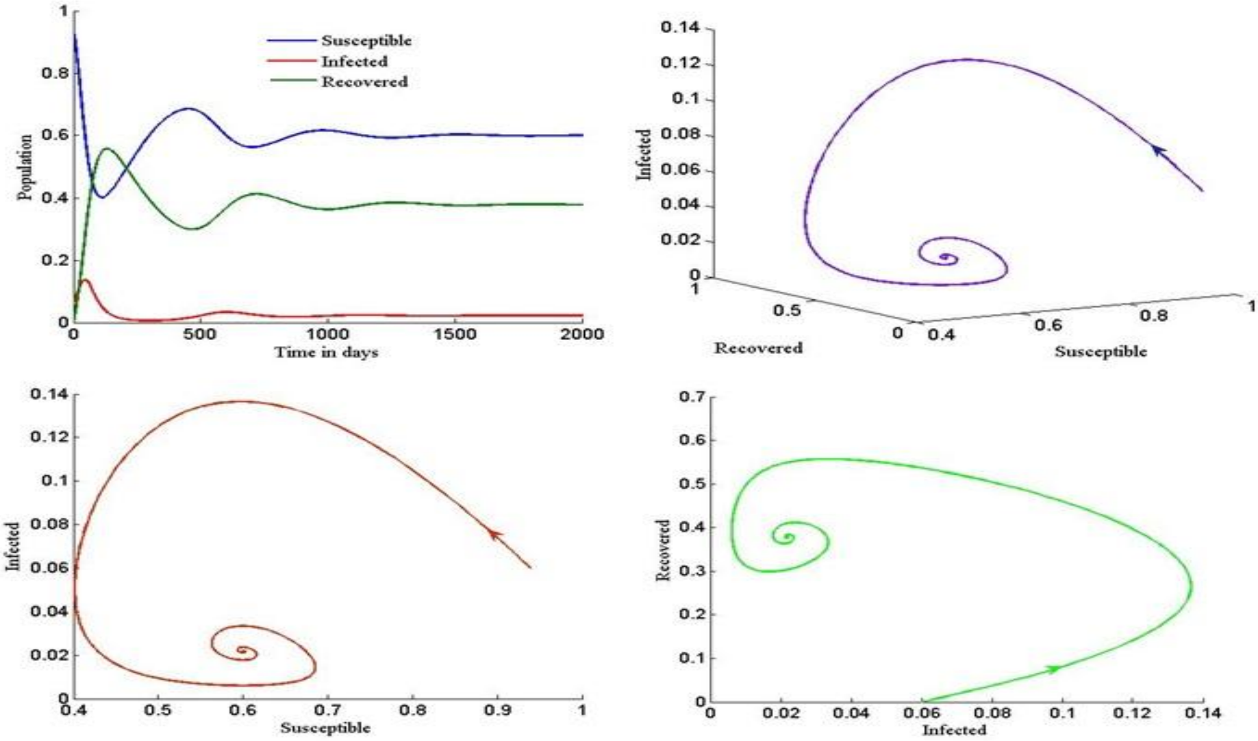

Fig. 4 Time plots and phase portraits of EE point E_1 with stability $\Re_0>1$

TABLE II. PARAMETERS FOR NUMERICAL SIMULATIONS OF THE MODEL (2).

Parameters	P	β	δ	μ	h	α	E_1
Case 3	0.05	0.8	0.45	0.03	0.1	0.9	$(0.6, 0.0219, 0.3781)$
Case 4	0.01	0.9	0.32	0.03	0.1	0.9	$(0.3889, 0.0515, 0.5596)$
Case 5	0.01	0.9	0.405	0.03	0.1	0.9	$(0.4833, 0.0349, 0.4817)$

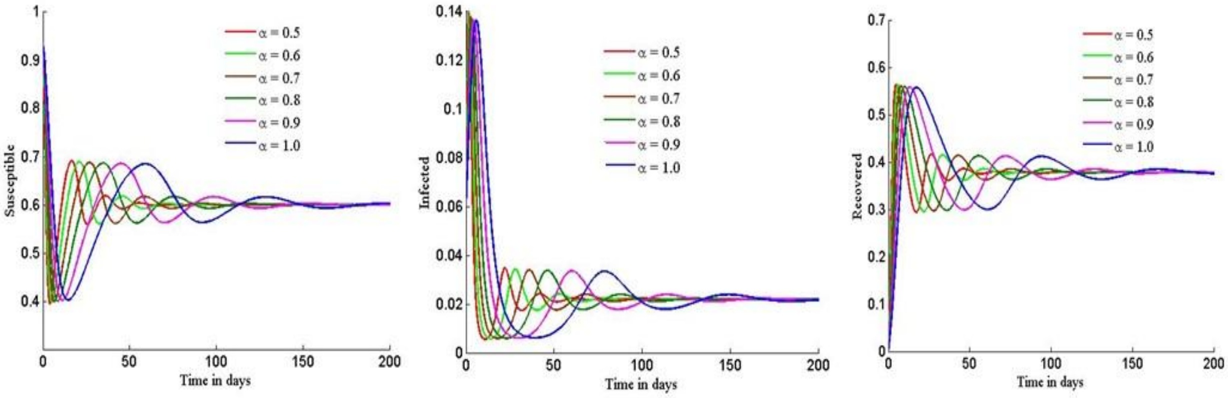

Fig. 5 Time plots of EE point E_1 for different fractional order $\alpha \in (0.5, 1.0]$ with stability $\Re_0 > 1$

In case 4, from $P(1) = 0.0000010920 > 0$, $P(-1) = -7.9449 < 0$, $a_{33} = -0.9864 < 1$ and $\Re_0 = 2.5457 > 1$ then the EE point $E_1 = (0.3889, 0.0515, 0.5596)$ of the model (2) is LAS (see Fig. 6 & 7).

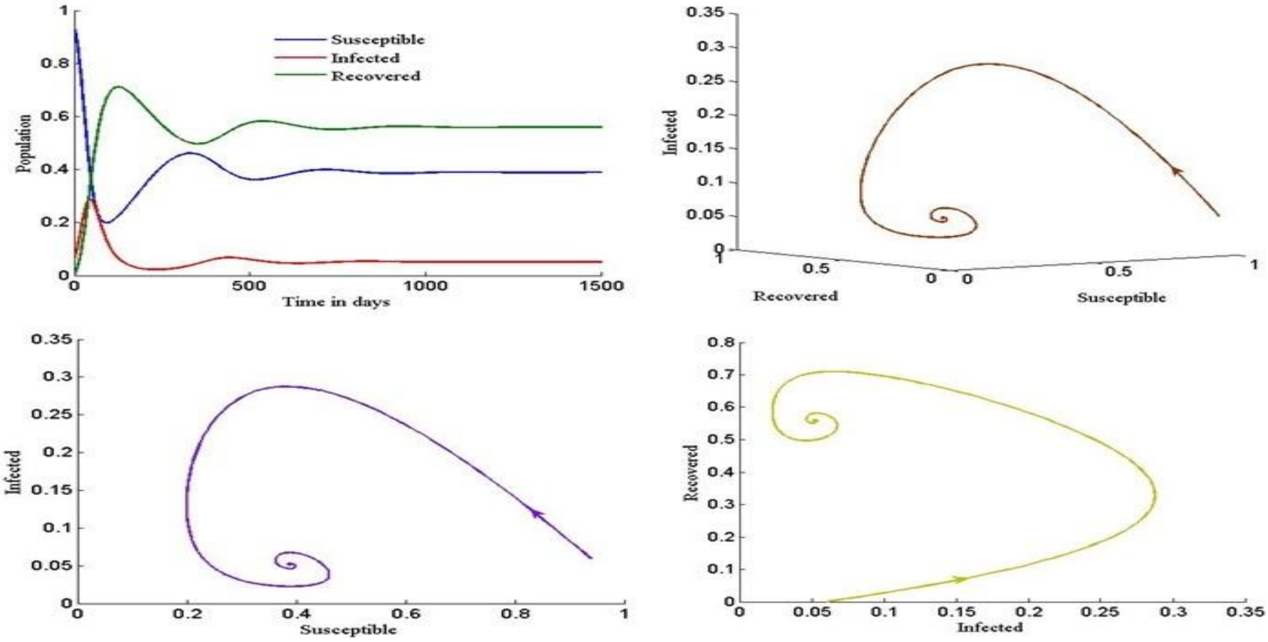

Fig. 6 Time plots and phase portraits of EE point E_1 with stability $\Re_0 > 1$

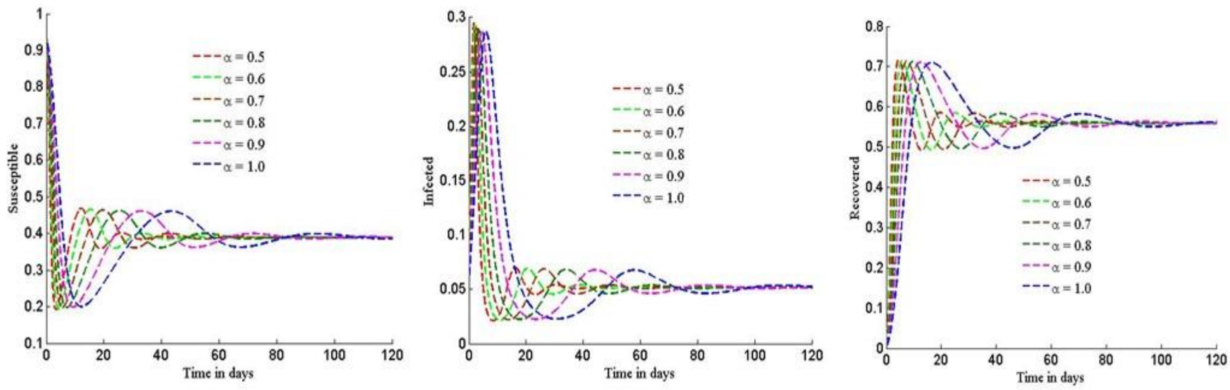

Fig.7 Time plots of EE point E_1 for different fractional order $\alpha \in (0.5, 1.0]$ with stability $\Re_0 > 1$

In case 5, from $P(1) = 0.00000092045 > 0$, $P(-1) = -7.9526 < 0$, $a_{33} = -0.9883 < 1$ and $\Re_0 = 2.0483 > 1$ then the EE point $E_1 = (0.4833, 0.0349, 0.4817)$ of the model (2) is LAS (see Fig. 8 & 9).

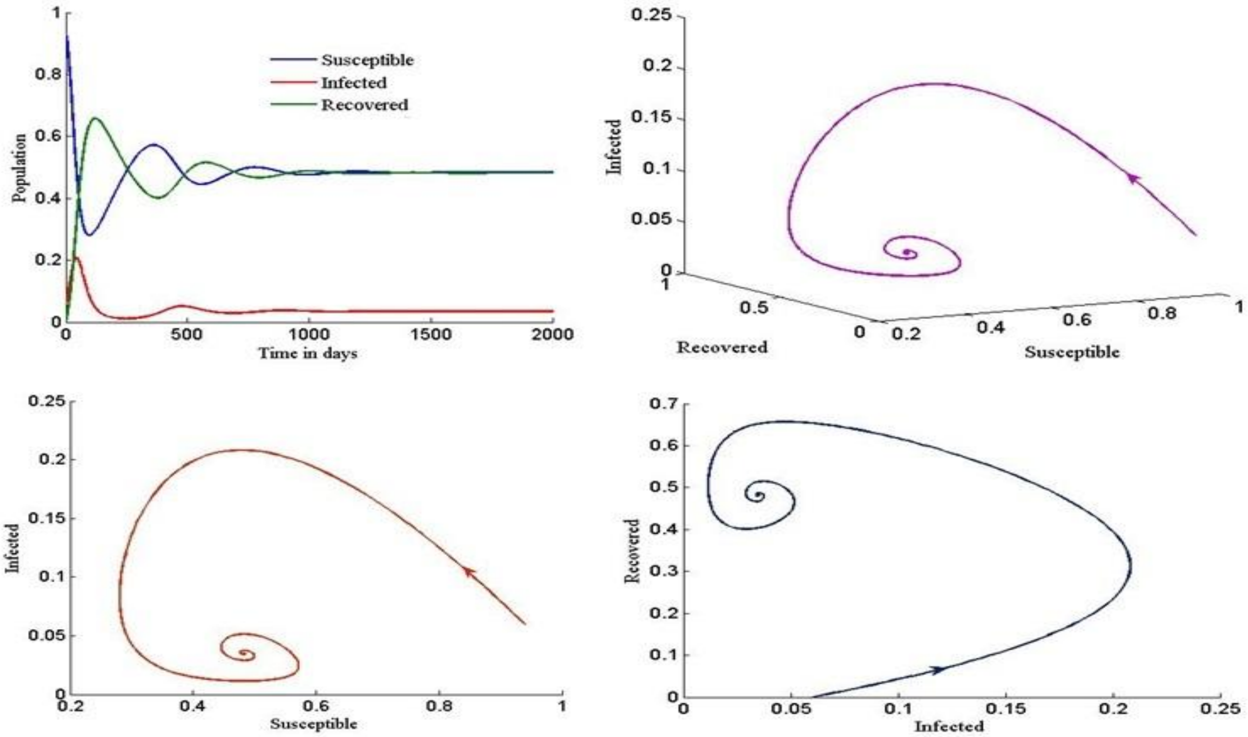

Fig. 8 Time plots and phase portraits of EE point E_1 with stability $\Re_0 > 1$

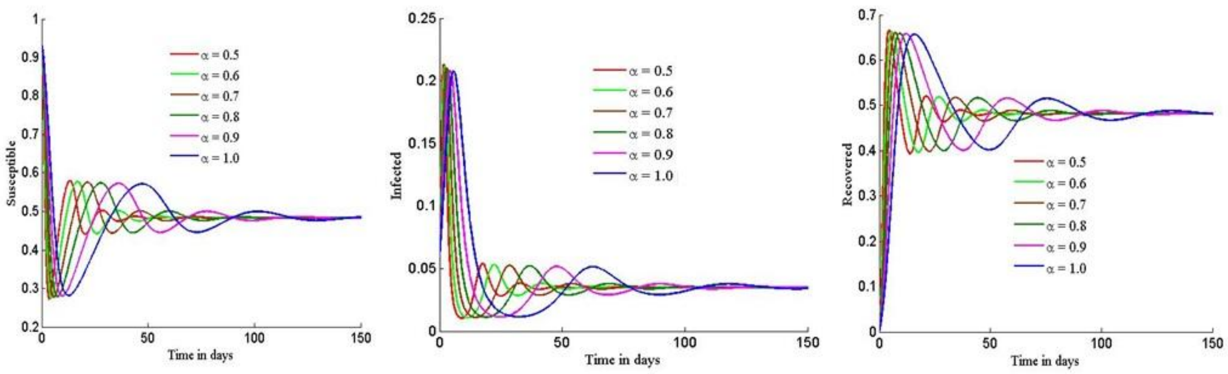

Fig. 9 Time plots of EE point E_1 for different fractional order $\alpha \in (0.5, 1.0]$ with stability $\Re_0 > 1$

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