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Parametric Successive Over Relaxation (PSOR) Method

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Abstract:

In this paper, we propose PSOR method for solving linear systems of equations by introducing a parameter 'α' in the SOR method and show that the spectral radius of the iterative matrix is same as that of the SOR matrix for any choice of 'α'. Few numerical examples are considered to justify our results.

Key words: Iterative methods, Iteration matrices, linear systems, SOR method.

1. Introduction

In many scientific and engineering applications one often come across with the problems of finding the solutions of linear system of equations of the form

$$AX = b \tag{1.1}$$

where A is a non singular matrix of order $n \times n$ and X and b are n-dimensional unknown and known column vectors respectively.

Without any loss of generality, if we let

$$A = I - L - U \tag{1.2}$$

where I is an unit matrix, $-L$ and $-U$ are strictly lower and upper triangular parts of A. Then, the system (1.1) takes the form

$$(I - L - U)X = b \tag{1.3}$$

The well known SOR method for the solution of (1.3) is given by

$$(I - \omega L)X^{(n+1)} = \left[(1-\omega)I + \omega U\right]X^{(n)} + \omega b$$

$$(n = 0,1,2...) \tag{1.4}$$

whose iteration matrix is

$$S_\omega = (I - \omega L)^{-1}\{(1-\omega)I + \omega U\} \tag{1.5}$$

The KSOR method introduced by Youssef[ ] is

$$\left[(1+\omega)I - \omega L\right]X^{(n+1)} = \left[I + \omega U\right]X^{(n)} + \omega b \tag{1.6}$$

$$(n = 0,1,2...)$$

whose iteration matrix is

$$K_\omega = \left[(1+\omega)I - \omega L\right]^{-1}\{I + \omega U\} \tag{1.7}$$

We introduce PSOR method and obtain the relation between the eigen values of PSOR iteration matrix and those of the Jacobi iteration matrix in section 2.

In section 3, we hall show that spectral radius of PSOR iteration matrix is independent of the parameter 'α' for a given optimal ω and vice-versa. Few examples are considered in the concluding section.

2. Parametric SOR method

By introducing a parameter $'\alpha'$ in the method (1.4), the PSOR method can be defined as

$$\left[(1+\omega-\alpha\omega)I-\omega L\right]X^{(n+1)}=\left[(1-\alpha\omega)I+\omega U\right]X^{(n)}+\omega b,\quad (\omega\neq 0) \tag{2.1}$$

which is completely consistent for any choice of $'\alpha'$.

It is note that for the choices of $\alpha=1$ and $\alpha=0$,the method(2.1) takes the forms (1.4) and (1.6) respectively.

The iteration matrix of the method (2.1) is

$$T_{\alpha,\omega}=\left[(1+\omega-\alpha\omega)I-\omega L\right]^{-1}\left\{(1-\alpha\omega)I+\omega U\right\} \tag{2.2}$$

Theorem 2.1:

The characteristic equation of iteration matrix $T_{\alpha,\omega}$ is

$$\left|\left[\lambda(1+\omega-\alpha\omega)+\alpha\omega-1\right]I-\omega(\lambda L+U)\right|=0 \tag{2.3}$$

Proof:

$$\text{Let } \left|T_{\alpha,\omega}-\lambda I\right|=0 \text{ or } \left|\lambda I-T_{\alpha,\omega}\right|=0$$

Then,

$$\left|\lambda I-\left[(1+\omega-\alpha\omega)I-\omega L\right]^{-1}\left\{(1-\alpha\omega)I+\omega U\right\}\right|=0$$

$$\Rightarrow\left|\lambda\left[(1+\omega-\alpha\omega)I-\omega L\right]-\left[(1-\alpha\omega)I+\omega U\right]\right|=0$$

which gives us

$$\left|\left[\lambda(1+\omega-\alpha\omega)+\alpha\omega-1\right]I-\omega(\lambda L+U)\right|=0$$

is the characteristic equation of $T_{\alpha,\omega}$

Theorem 2.2: The eigen values of the Jacobi iteration matrix $J=L+U$ and the PSOR iteration matrix $T_{\alpha,\omega}$ are connected by the relation

$$\left[\lambda(1+\omega-\alpha\omega)+\alpha\omega-1\right]^2=\lambda\omega^2\mu^2 \tag{2.4}$$

Proof: let λ and μ be the eigen values of $T_{\alpha,\omega}$ and J then by theorem 1, we have

$$\left|\left[\lambda(1+\omega-\alpha\omega)+\alpha\omega-1\right]I-\omega(\lambda L+U)\right|=0$$

$$\Rightarrow\left|\frac{1}{\omega}\left[\lambda(1+\omega-\alpha\omega)+\alpha\omega-1\right]I-(\lambda L+U)\right|=0,\quad (\omega\neq 0) \tag{2.5}$$

As discussed by Young [2,3] and Varga [1] from (2.5) ,we can have

$$\frac{1}{\omega}\left[\lambda(1+\omega-\alpha\omega)+\alpha\omega-1\right]=\lambda^{\frac{1}{2}}\mu,$$

Or

$\left[\lambda(1+\omega-\alpha\omega)+\alpha\omega-1\right]^2=\lambda\omega^2\mu^2$ is the required relation between λ and μ

3. Spectral radius of $T_{\alpha,\omega}$

Solving the equation (2.4) for λ, we obtain

$$\lambda=\frac{\omega^2\mu^2-2(\alpha\omega-1)(1+\omega-\alpha\omega)\pm\sqrt{\Delta}}{2(1+\omega-\alpha\omega)^2} \tag{3.1}$$

Where,

$$\Delta=\omega^4\mu^4-4\omega^2\mu^2(\alpha\omega-1)(1+\omega-\alpha\omega) \tag{3.2}$$

after simple manipulations, we have

$$\Delta = \omega^2 \mu^2 \left[\left(4\alpha^2 - 4\alpha + \mu^2\right)\omega^2 - \left(8\alpha - 4\right)\omega + 4 \right] \tag{3.3}$$

For the choice of $\omega = \omega^*$

$$\omega^* = \frac{2}{\left(2\alpha - 1\right) + \sqrt{1 - \bar{\mu}^2}} \tag{3.4}$$

where $\bar{\mu}$ be the maximum eigen value of Jacobi iteration matrix in magnitude,
the spectral radius of $T_{\alpha,\omega}$ happens to be

$$S\left(T_{\alpha,\omega}\right) = \frac{\alpha^* \omega - 1}{1 + \omega - \alpha^* \omega} \tag{3.5}$$

$$= \frac{1 - \sqrt{1 - \bar{\mu}^2}}{1 + \sqrt{1 - \bar{\mu}^2}}$$

$$= \frac{\bar{\mu}^2}{\left(1 + \sqrt{1 - \bar{\mu}^2}\right)^2} \tag{3.6}$$

which is independent of $'\alpha'$ and same as the spectral radius of SOR iteration matrix.
The discriminant Δ of (3.3) can also be written as

$$\Delta = \omega^2 \mu^2 \left[\omega^2 \alpha^2 - \left(\omega^2 + 2\omega\right)\alpha + \left(\frac{\omega^2 \mu^2}{4} + \omega + 1\right) \right] \tag{3.7}$$

For the choice of $\alpha = \alpha^*$ as

$$\alpha^* = \frac{1}{\omega} + \frac{1}{2\left(1 + \sqrt{1 - \bar{\mu}^2}\right)}, \tag{3.8}$$

the spectral radius of $T_{\alpha,\omega}$ happens to be

$$S\left(T_{\alpha,\omega}\right) = \frac{\alpha \omega^* - 1}{1 + \omega^* - \alpha \omega^*} \tag{3.9}$$

$$= \frac{1 - \sqrt{1 - \bar{\mu}^2}}{1 + \sqrt{1 - \bar{\mu}^2}} \tag{3.10}$$

which is independent of ω and same as $S\left(S_\omega\right)$

It is now interesting to note that $S\left(T_{\alpha,\omega}\right)$ will be unaltered if $\omega = \omega^*$ for any $'\alpha'$ and if $\alpha = \alpha^*$ for any ω

4. Numerical examples

We consider the example as studied by Youssef [4] i.e.,

$$\left.\begin{aligned} x - 0.5y &= 0.5 \\ -0.5x + y &= 0.5 \end{aligned}\right\} \tag{4.1}$$

The spectral radius of Jacobi iteration matrix is

$$S(L+U) = S\left(\begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix}\right) = \bar{\mu} = 0.5 \tag{4.2}$$

With this value of $\bar{\mu}$, we obtained

$$\omega^* = \frac{2}{2\alpha - 0.133974596} \tag{4.3}$$

and

$$\alpha^* = \frac{1}{\omega} + 0.267949192 \tag{4.4}$$

Applying PSOR method for the solution of (4.1) (whose exact solution is $x = y = 1$) taking the initial guess for x and y as $x^{(0)} = 0, y^{(0)} = 0$ and tabulate the computational results taking $\omega = \omega^*$ for different values of α and also taking $\alpha = \alpha^*$ for various choices of ω.

The errors obtained in each iteration using the formula

$$\text{Error} = \sqrt{(\text{Exact } x\text{-App.}x)^2 + (\text{Exact } y\text{-App.}y)^2} \tag{4.5}$$

are also tabulated.

for $\alpha = 0$, by (4.3) $\omega^* = -14.9282032$ (KSOR method)			
n	x	y	error
0	0.53589838	0.82308546	0.49667803
1	0.93851278	0.97975099	0.64735618e-1
2	0.99356317	0.99800433	0.67390987e-2
3	0.99939267	0.99981781	0.63407134e-3
4	0.99994597	0.99998413	0.56312383e-4
5	0.99999537	0.99999866	0.4817687e-5
6	0.99999961	0.99999989	0.40151471e-6
7	0.99999997	0.99999999	0.3282099e-7
8	1	1	0.2643172e-8

Table (4.1)

For $\alpha = 1$, by (4.3) $\omega^* = 1.07179677$ (SOR method)			
n	x	y	error
0	0.53589838	0.82308546	0.49667803
1	0.93851278	0.97975099	0.64735618e-1
2	0.99356317	0.99800433	0.67390987e-2
3	0.99939267	0.99981781	0.63407134e-3
4	0.99994597	0.99998413	0.56312383e-4
5	0.99999537	0.99999866	0.4817687e-5
6	0.99999961	0.99999989	0.40151471e-6
7	0.99999997	0.99999999	0.3282099e-7
8	1	1	0.26431718e-8

Table (4.2)

For $\alpha = -10$, by (4.3) $\omega^* = -0.99334584e-1$			
n	x	y	error
0	0.53589838	0.82308546	0.49667803
1	0.93851278	0.97975099	0.64735618e-1
2	0.99356317	0.99800433	0.67390987e-2
3	0.99939267	0.99981781	0.63407134e-3
4	0.99994597	0.99998413	0.56312383e-4
5	0.99999537	0.99999866	0.4817687e-5
6	0.99999961	0.99999989	0.40151471e-6
7	0.99999997	0.99999999	0.3282099e-7
8	1	1	0.26431716e-8

Table (4.3)

For $\omega = 1$, by (4.4) $\alpha^* = 1.0669873$			
n	x	y	error
0	0.53589838	0.82308546	0.49667803
1	0.93851278	0.97975099	0.64735618e-1
2	0.99356317	0.99800433	0.67390987e-2
3	0.99939267	0.99981781	0.63407134e-3
4	0.99994597	0.99998413	0.56312383e-4
5	0.99999537	0.99999866	0.4817687e-5
6	0.99999961	0.99999989	0.40151471e-6
7	0.99999997	0.99999999	0.3282099e-7
8	1	1	0.26431718e-8

Table (4.4)

For $\omega = -1$, by (4.5) $\alpha^* = -0.9330127$			
n	x	y	error
0	0.53589838	0.82308546	0.49667803
1	0.93851278	0.97975099	0.64735618e-1
2	0.99356317	0.99800433	0.67390987e-2
3	0.99939267	0.99981781	0.63407134e-3
4	0.99994597	0.99998413	0.56312383e-4
5	0.99999537	0.99999866	0.4817687e-5
6	0.99999961	0.99999989	0.40151471e-6
7	0.99999997	0.99999999	0.3282099e-7
8	1	1	0.26431717e-8

Table(4.5)

From the above tabulated results, it is evident that the rate of convergence happens to be the same when $\omega = \omega^*$ for any α and $\alpha = \alpha^*$ for any ω.

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