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QUADRATIC HARVESTING IN A DISCRETE PREY-PREDATOR MODEL WITH SCAVENGER

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Abstract - The work is related to dynamical nature of a discrete time three species prey-predator-scavenger model in the presence of quadratic harvesting on predator population. We investigate existence and parametric conditions for local stability of positive equilibrium point of this model. Moreover, it is also proved that the system under goes Neimark-Sacker (NS) and Period-Doubling bifurcation (PDB) at certain parametric values for positive equilibrium point with the help of an explicit criterion for NS and PDB. The trajectories and phase plane diagrams are plotted for biologically meaningful sets of parameter values. Also bifurcation diagram are shown for selected range of growth parameter. Finally, a numerical example is provided for justifying the validity of the theoretical analysis and visualizes the model with and without harvesting on predator.

Keywords – Discrete Time, Equilibria, Scavenger, Harvesting, Stability, Limit Cycles, Bifurcation Diagrams.

1. INTRODUCTION

Mathematical models are of importance in examining the complex dynamics of interacting populations. Lotka and Volterra introduced the first mathematical model which described the interaction of populations [2]. Since then several models appeared by including more species and many types of functional responses [9, 6] making the classical model more realistic. We consider three species model which includes prey, predator and scavengers in an ecosystem. Scavengers play an important role in food web by consuming decaying dead animal. These species are capable of breaking down the organic material, which includes bodies of dead animals, and recycling it into the ecosystems as nutrients [3, 4, 1].

2. EQUILIBRIA OF THE DISCRETE MODEL

Consider the following discrete time predator-prey-scavenger model:

$$\begin{aligned} x(t+1) &= x(t)\left(1-rx(t)\right)+x(t)(1-y(t)-z(t)) \\ y(t+1) &= y(t)+(x(t)-\tau)y(t) \\ z(t+1) &= z(t)(1+ax(t))+\rho y(t)z(t)-\nu z(t) \end{aligned} \tag{1}$$

where r is the growth rate associated with prey x , τ & ν are the natural death rate of scavenger z & predator y , a is the rate of change in the scavenger involving a prey population, ρ is the rate of change in the scavenger involving a predator. When the prey-predator-scavenger system (1) subject to the quadratic harvesting of predator, the model becomes:

$$\begin{aligned} x(t+1) &= x(t)\left(1-rx(t)\right)+x(t)(1-y(t)-z(t)) \\ y(t+1) &= y(t)+(x(t)-\tau)y(t)-\eta y^2(t) \\ z(t+1) &= z(t)(1+ax(t))+\rho y(t)z(t)-\nu z(t)-\mu z^2(t) \end{aligned} \tag{2}$$

Here η and μ are both carrying capacities of scavenger and predator respectively but the term ηy^2 and μz^2 are quadratic harvesting of scavenger and predator population. The above system (2) has five equilibrium points: (i) The trivial equilibrium $E_0=(0,0,0)$ and the predator free equilibrium $E_1=\left(\frac{1}{r},0,0\right)$, which are always feasible. (ii) The boundary equilibrium in xy -plane is given by $E_{xy}=(\bar{x},\bar{y},0)=\left(\frac{\tau+\eta}{r\eta+1},\frac{1-r\tau}{r\eta+1},0\right)$, which is feasible when $r\tau<1$. (iii) The second boundary equilibrium in xz -plane is given by $E_{xz}=(\hat{x},0,\hat{z})=\left(\frac{\mu+\nu}{a+r\mu},0,\frac{a-r\nu}{a+r\mu}\right)$, which is feasible when $r\nu<a$. (iv) The positive equilibrium is $E_*=(x_*,y_*,z_*)$ with $x_*=\frac{(\rho+\mu)\tau+(\nu+\mu)\eta}{(a\eta+\rho)+(r\eta+1)\mu}$, $y_*=\frac{(\mu+\nu)-(a+r\mu)\tau}{(a\eta+\rho)+(r\eta+1)\mu}$,

$z_* = \dfrac{(a-r\nu)\eta + (\rho-\nu) + (a-r\rho)\tau}{(a\eta+\rho)+(r\eta+1)\mu}$. This equilibrium exist if $\mu+\nu > (a+r\mu)\tau$, $a(\tau+\eta)+\rho > r\rho\tau + \nu(1+r\eta)$.

3. DYNAMICAL NATURE OF THE MODEL

This section discusses the local behavior of the system (2) in presence of quadratic harvesting around each equilibrium point. The variational matrix of system (2) is

$$J(x,y,z) = \begin{bmatrix} 2(1-rx)-y-z & -x & -x \\ y & 1+x-\tau-2\eta y & 0 \\ az & \rho z & 1+ax+\rho y-\nu-2\mu z \end{bmatrix} \tag{3}$$

3.1. Stability of E_0: The eigenvalues of the variational matrix J for the system evaluated at the trivial point E_0 are $\lambda_1 = 2$, $\lambda_2 = 1-\tau$ and $\lambda_3 = 1-\nu$. Thus E_0 is saddle when $|\lambda_1|>1$, $0<\tau<2$ and $0<\nu<2$.

3.2. Stability of $E_1 = \left(\frac{1}{r},0,0\right)$: The eigenvalues of J for the system evaluated at the axial equilibrium point E_1 are $\lambda_1 = 0$, $\lambda_2 = 1+\frac{1}{r}-\tau$ and $\lambda_3 = 1+\frac{a}{r}-\nu$. Hence E_1 is locally asymptotically stable if it satisfies the conditions $|\lambda_1|<1$, $\frac{1}{r}<\tau<2+\frac{1}{r}$ and $\frac{a}{r}<\nu<2+\frac{a}{r}$. Otherwise it is locally unstable.

3.3. Stability of $E_{xy} = (\bar{x},\bar{y},0)$: The eigenvalues of the variational matrix J evaluated at the first boundary equilibrium point E_{xy} are $\lambda_1 = 1+a\bar{x}+\rho\bar{y}-\nu$ and $\lambda_{2,3} = \dfrac{(1+\bar{y}-\eta\bar{y}) \pm \sqrt{(1-\bar{y})^2+\eta\bar{y}(\eta\bar{y}-2+2\bar{y})-4\bar{x}\,\bar{y}}}{2}$. Hence E_{xy} is locally asymptotically stable when $a\bar{x}+\rho\bar{y}<\nu<2+a\bar{x}+\rho\bar{y}$ and $\eta(\bar{y}-1)-\bar{x}<0$. The equilibrium point E_{xy} becomes saddle if $\nu>2+a\bar{x}+\rho\bar{y}$ and $\eta(\bar{y}-1)-\bar{x}<0$.

3.4. Stability of $E_{xz} = (\hat{x},0,\hat{z})$: The eigenvalues of the variational matrix J evaluated at the second boundary equilibrium point E_{xz} are $\lambda_1 = 1-\tau+\hat{x}$ and $\lambda_{2,3} = \dfrac{(1+\hat{z}-\mu\hat{z}) \pm \sqrt{(1-\hat{z})^2+\mu\hat{z}(\mu\hat{z}-2+2\hat{z})-4a\hat{x}\hat{z}}}{2}$. Hence E_{xz} is locally asymptotically stable when $\hat{x}<\tau<2+\hat{x}$ and $\mu(\hat{z}-1)-a\hat{x}<0$. The equilibrium point E_{xz} becomes locally unstable if $\tau>2+\hat{x}$ and $\mu(\hat{z}-1)-a\hat{x}>0$.

4. LOCAL STABILITY AND BIFURCATION

Now, we discuss the stability and the conditions for the existence of NSB and PDB at the positive equilibrium point E_ of the scavenger system (2). Recently, many authors have discussed similar type of bifurcation for discrete time dynamical systems. The variational matrix (3) at $E_* = (x_*, y_*, z_*)$ has the form

$$J(E_*) = \begin{bmatrix} -\beta_1 & -x_* & -x_* \\ y_* & -\beta_2 & 0 \\ az_* & \rho z_* & -\beta_3 \end{bmatrix}$$

The characteristic equation of $J(E_*)$ is $\lambda^3+S_1\lambda^2+S_2\lambda+S_3=0$ (4)

where $S_1=\beta_1+\beta_2+\beta_3$, $S_2=\beta_1\beta_2+\beta_2\beta_3+\beta_3\beta_1+ax_*z_*+x_*y_*$, $S_3=\beta_1\beta_2\beta_3+\beta_3x_*y_*+\rho x_*y_*z_*+\beta_2ax_*z_*$ (5)

such that $\beta_1=2(rx_*-1)+y_*+z_*$, $\beta_2=\tau-x_*+2\eta y_*-1$ and $\beta_3=\nu-ax_*-\rho y_*+2\mu z_*-1$. By Routh-Hurwitz criterion, the positive equilibrium point E_* is locally asymptotically stable if and only if $S_1>0$, $S_3>0$ and $\beta_1^2(\beta_2+\beta_3)+\beta_2^2(\beta_3+\beta_1)+\beta_3^2(\beta_1+\beta_2)+(\beta_1+\beta_2)x_*y_*+(\beta_1+\beta_3)ax_*z_*+2\beta_1\beta_2\beta_3>\rho x_*y_*z_*$

4.1. Neimark-Sacker Bifurcation: To study the NSB of the system (2), we need the explicit criterion of Hopf bifurcation [7] is useful [10].

Theorem 1: The positive equilibrium point E_* of the system (2) undergoes NSB for $\mu+\nu>(a+r\mu)\tau$, $a(\tau+\eta)+\rho>r\rho\tau+\nu(1+r\eta)$ if the following conditions hold: $1-S_2+S_3(S_1-S_3)=0$, $1+S_2-S_3(S_1+S_3)>0$, $1+S_1+S_2+S_3>0$ and $1-S_1+S_2-S_3>0$, where S_1, S_2 and S_3 are given in (5).

Proof: For three dimensional system (n = 3) [5], we have the characteristic polynomial (4) of system (2) evaluated at the positive equilibrium point E_*. Thus we obtain the following equalities and inequalities: $\Delta_2^-(r)=1-S_2+S_3(S_1-S_3)=0$, $\Delta_2^+(r)=1+S_2-S_3(S_1+S_3)>0$, , $P_r(1)=1+S_1+S_2+S_3>0$ and $(-1)^3P_r(-1)=1-S_1+S_2-S_3>0$.

4.2. Period-Doubling Bifurcation: An explicit critical criterion for the existence of PDB is proposed for higher dimensional discrete time systems [8].

Theorem 2: The positive equilibrium point E_* of the system (2) undergoes PDB for $\mu+\nu>(a+r\mu)\tau$, $a(\tau+\eta)+\rho>r\rho\tau+\nu(1+r\eta)$ if the following conditions hold: $1-S_2+S_3(S_1-S_3)>0$, $1+S_2-S_3(S_1+S_3)>0$, $1\pm S_2>0$, $1+S_1+S_2+S_3>0$ and $-1+S_1-S_2+S_3=0$, where S_1, S_2 and S_3 are given in (5).

Proof: three dimensional system (n = 3) [5], we have the characteristic polynomial (4) of system (2) evaluated at the positive equilibrium point E_*. Then we obtain the following equalities and inequalities: $\Delta_2^-(r)=1-S_2+S_3(S_1-S_3)>0$, $\Delta_2^+(r)=1+S_2-S_3(S_1+S_3)>0$, $\Delta_1^\pm(r)=1\pm S_2$, $P_r(1)=1+S_1+S_2+S_3>0$ and $P_r(-1)=-1+S_1-S_2+S_3=0$.

5. NUMERICAL STUDY

The purpose of this section is to present phase trajectories, limit cycles and bifurcation diagrams for growth parameter r and harvesting parameter μ to illustrate the results obtained in the previous section. From the numerical results, in Figure (1) we use some parameters values like $r=2.271$, $\tau=0.049$, $\eta=0.035$, $a=0.0629$, $\rho=2.76$, $\nu=0.637$, $\mu=1.491$ with $(x,y,z)=(0.4,0.3,0.2)$. The positive equilibrium point is $E_*=(0.064,0.448,0.404)$ and the eigenvalues are

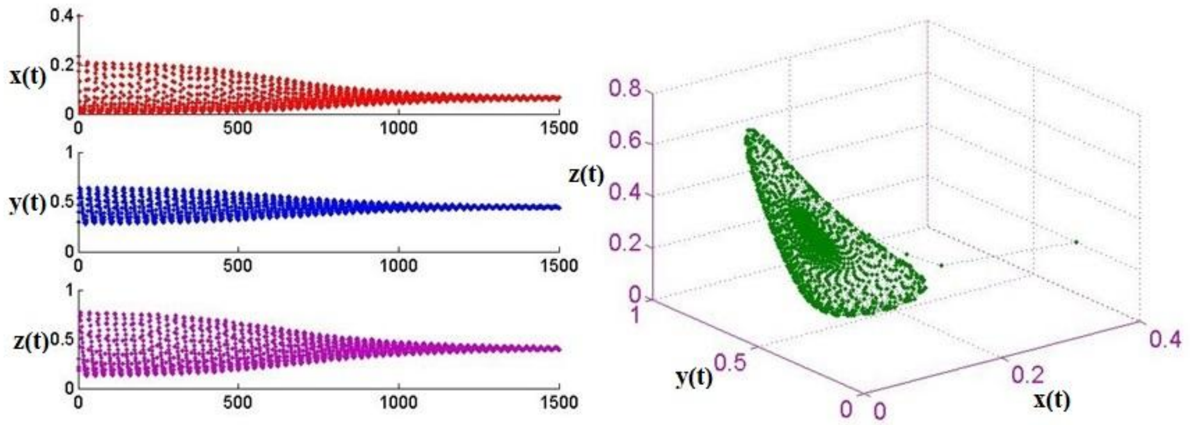

Figure 1: *The trajectory of system (2) converges to* E_*.

$\lambda_1=-0.6692$, $\lambda_2=0.9282$ and $\lambda_3=0.4516$ so that $|\lambda_{1,2,3}|<1$ which ensures stability of the system. In Figure (1), we observe that time plot is oscillatory but converges. Phase portrait spirals into stable equilibrium of the model (2). In this case the positive equilibrium point E_* is a sink and the system (2) is locally asymptotically stable.

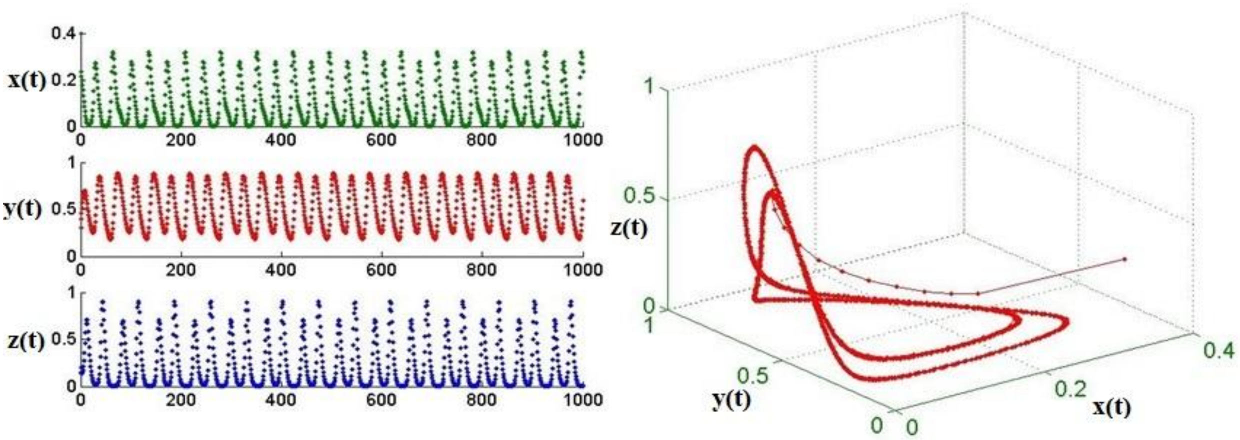

Figure 2: *The trajectory of system (2) is unstable.*

Whereas with $r=2.29$, $\tau=0.069$, $a=0.41$, $\rho=1.5173$, $\mu=0.391$ and keeping all other parameter values are same. Eigenvalues are $\lambda_1=1.3833$, $\lambda_{2,3}=0.9190\pm i0.0796$ so that $|\lambda_1|>1$ and $|\lambda_{2,3}|=0.9224<1$. In this case, the coexistence equilibrium point E_* is a saddle and the system (2) is unstable.

Figures (3) and (4) describe the approximate solutions x and y depend on the intrinsic growth parameter r are displayed in the figures below. We consider the parameter values are $\tau=0.049$, $\eta=0.035$, $a=0.0622$, $\rho=2.176$, $\nu=0.637$, $\mu=1.371$ with $(x,y,z)=(0.4,0.3,0.2)$. Other parameters will be (a) $r=2.28$, (b) $r=2.32$, (c) $r=2.34$, and (d) $r=2.36$. While with $\tau=0.099$, $\eta=0$, $a=0.699$, $\rho=2.017$, $\nu=1.296$, $\mu=0$ and the initial conditions are $(x,y,z)=(0.2,0.3,0.4)$. Other parameters will be (a) $r=2.181$, (b) $r=2.51$, (c) $r=2.61$, and (d) $r=2.92$. Figure (3) and (4) shows the phase trajectories of the system (2) and (1) involving both presence and absence of quadratic harvesting according to chosen parameter values and for various values of growth parameter r .

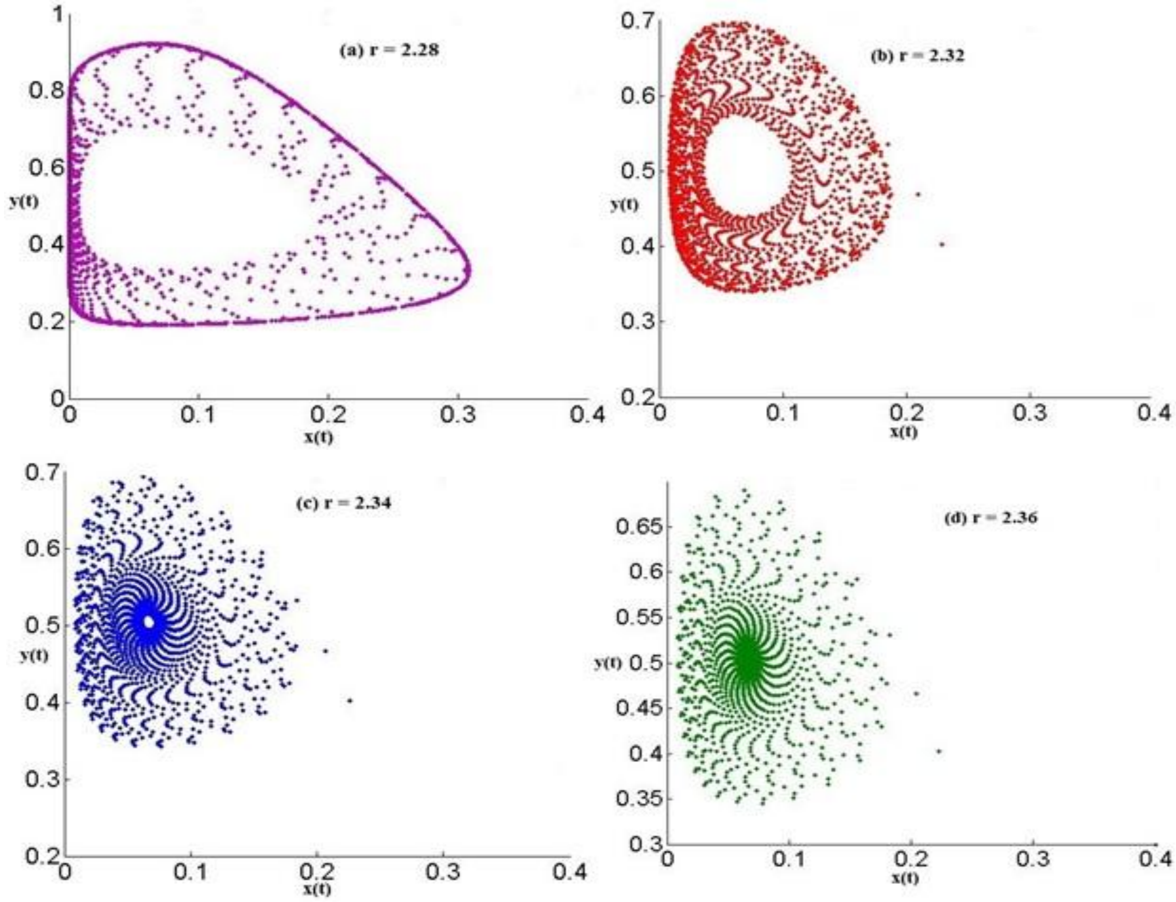

Figure 3: *The trajectories of system (2) moves from unstable to stable in the positive xy - octant.*

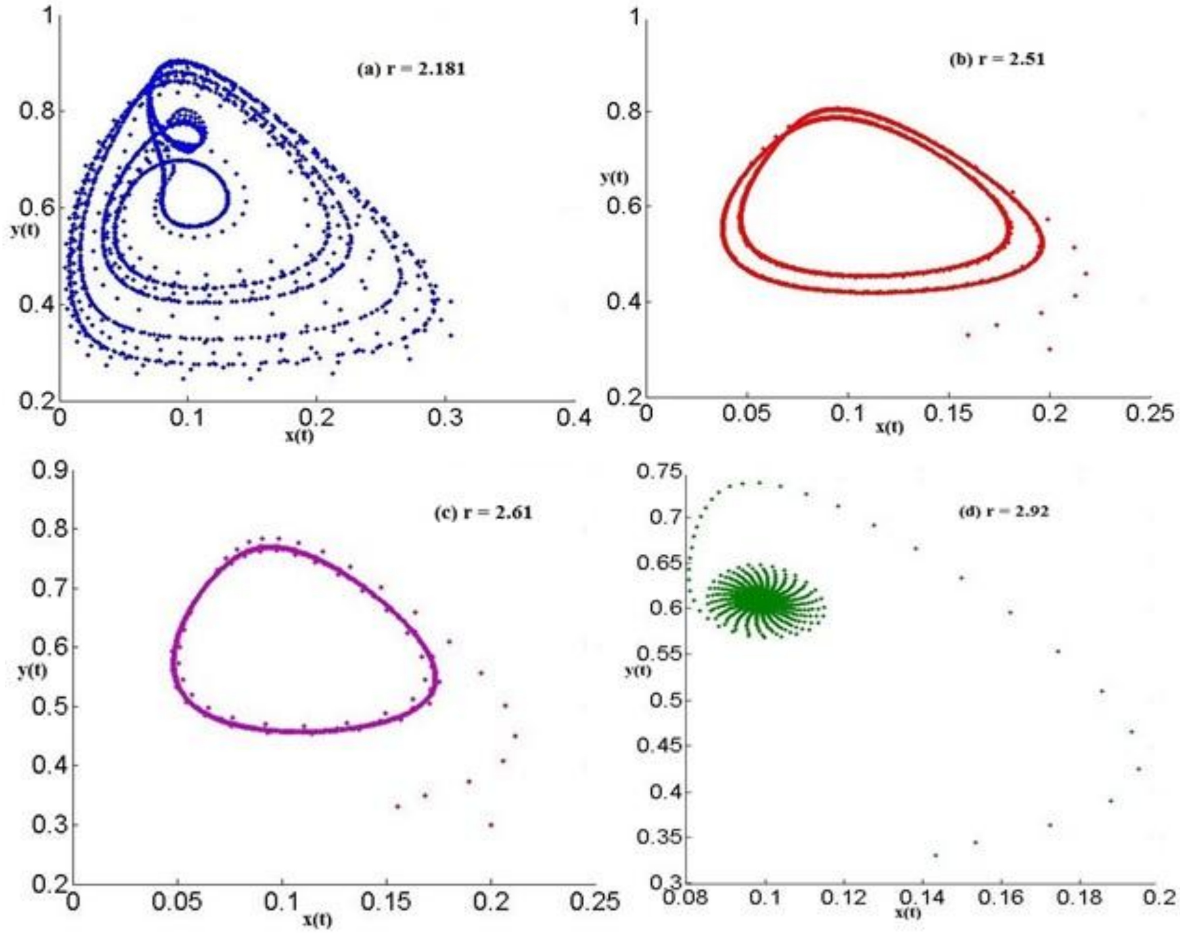

Figure 4: *Trajectories of system (1) moving from unstable to stable in the positive xy - plane.*

We can see that, whenever the value of r increases, then E_* moves from destabilized to stabilized and the trajectories spirals slowly inwards but does not approach a point. Finally settles down as a limit cycles (see Figure 3(a)-3(c), 4(a)-4(c)). When r increases at certain values, for example $r = 2.36$ and $r = 2.92$, the trajectories spirals much faster than approaches to an asymptotically stable of the system (1) and (2) (see Figure 3(d) & 4(d)). The bifurcation diagrams for growth rate r of the system (2) involving a presence of quadratic harvesting with $x = 0.4$, $y = 0.3$ & $z = 0.2$ as above and the selected parameter values $\tau = 1.39$, $\eta = 0.79$, $a = 2.89$, $\rho = 0.77$, $\nu = 0.75$, $\mu = 0.42$, $r \in [1,1.4]$ with step size $\Delta r = 0.001$ in the $(r - x)$ plane and the $(r - z)$ plane are given in Figure (5). We observe that the bifurcation diagrams of an equilibrium point E_{xz} for larger value of growth rate r of the prey and scavenger populations there is no possibility of chaotic dynamics of the above system(2). But for smaller value of r the systems becomes chaotic.

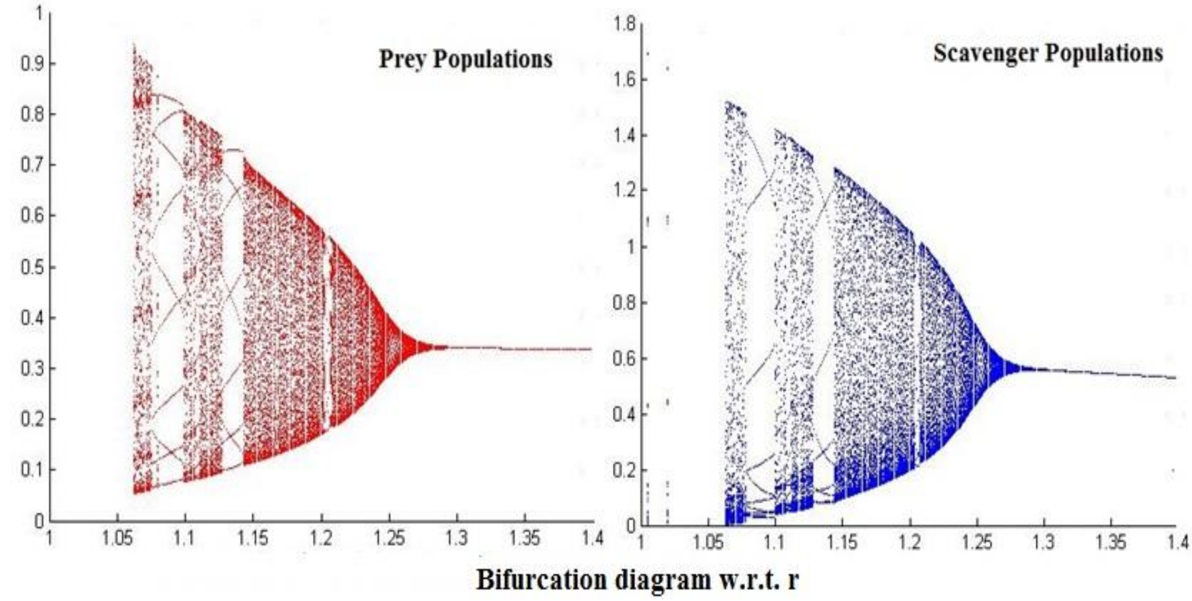

Figure 5: *Bifurcation diagram for prey & Scavenger populations with growth rate parameter r of system (2).*

The phase portraits which are associated with Figure 5(a - b) are disposed in Figure (6), which clearly depicts the process of how a smooth invariant circle bifurcates from the stable $(0.342, 0, 0.568)$. When $r < 1.26$ there appears a

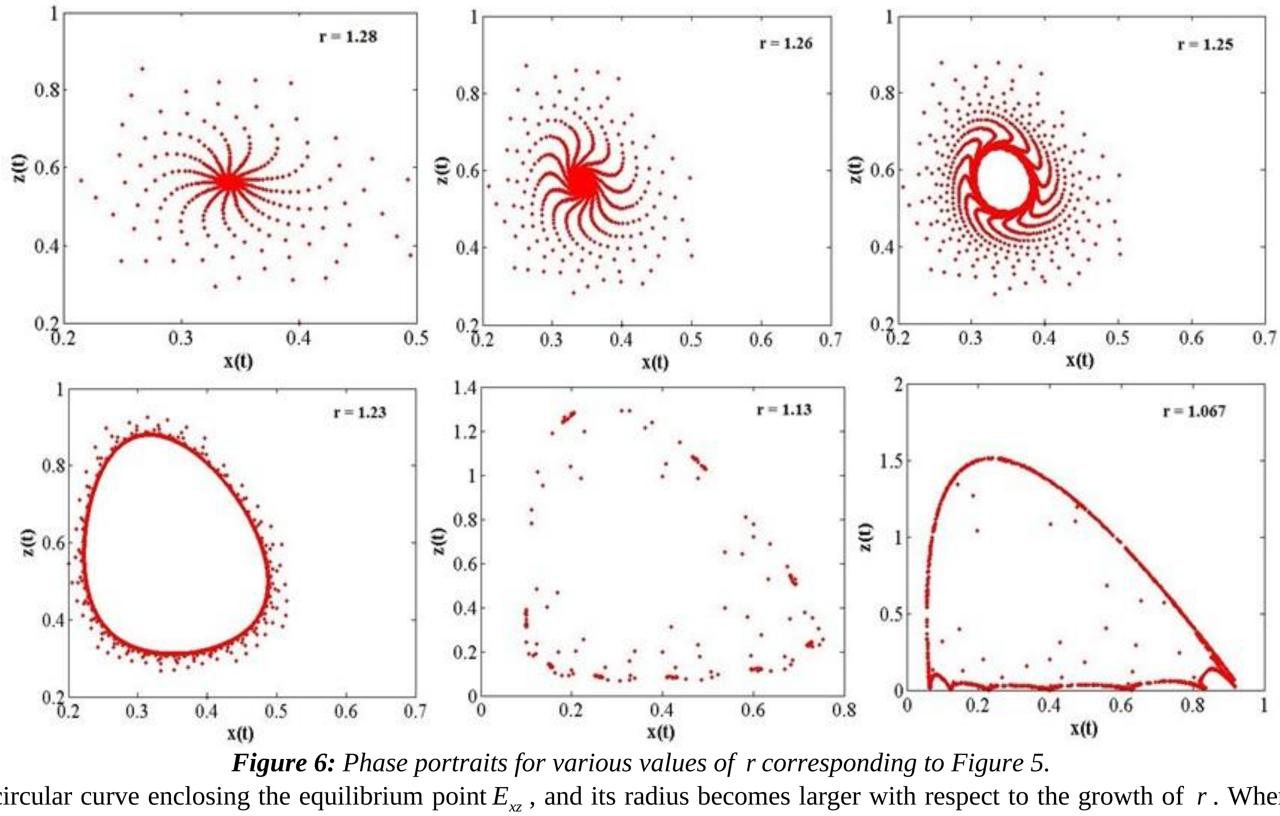

Figure 6: *Phase portraits for various values of r corresponding to Figure 5.*

circular curve enclosing the equilibrium point E_{xz}, and its radius becomes larger with respect to the growth of r. When r increases at certain value, for example at $r = 1.15$, the circle disappears and quasi-periodic orbits lead to chaos.

5.1. Sensitive Dependence On Initial Values: The sensitivity to initial conditions is a characteristic of chaos. In order to demonstrate the sensitivity to initial values of the scavenger system (2), we compute four orbits for both prey and scavenger populations with the initial values (x_0, y_0, z_0), $(x_0 + 0.0001, y_0, z_0)$ and $(x_0, y_0, z_0 + 0.0001)$ respectively. The compositional results are shown in Figure (7) & (8). From these figures it is clearly observe that, at the beginning, the time plots are indistinguishable but after a number of iterations, the difference between them builds up rapidly.

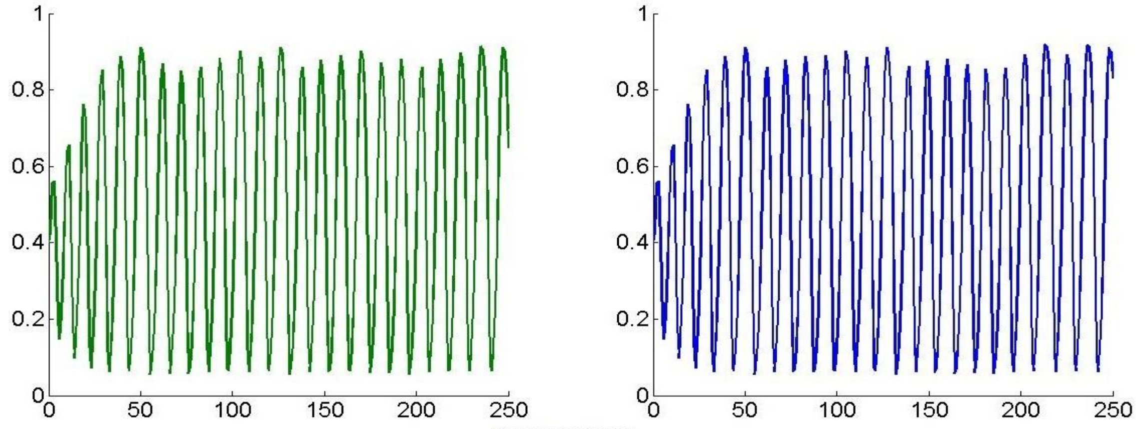

Figure 7: *Time plots* x_n *corresponding to the initial conditions* $(0.4, 0.3, 0.2) \& (0.4001, 0.3, 0.2)$.

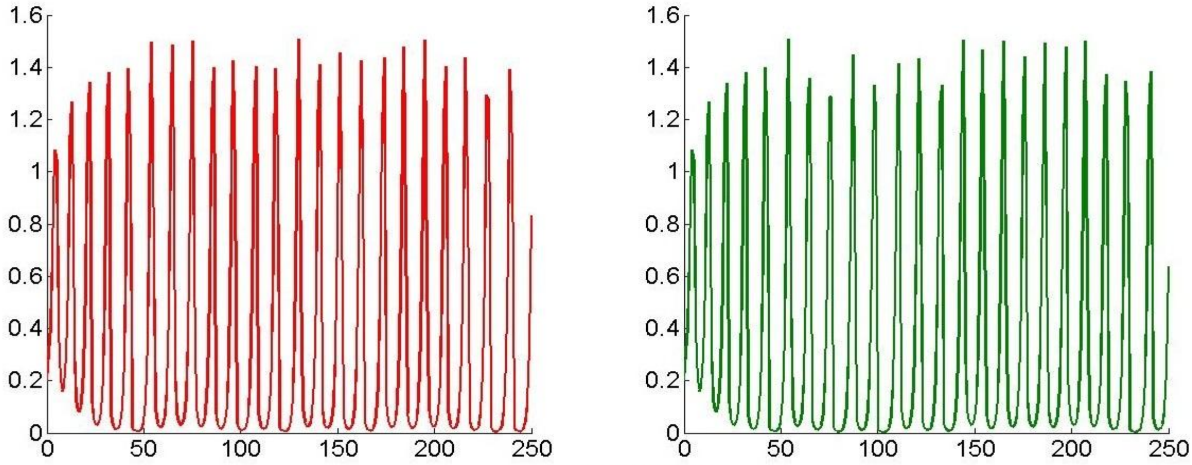

Figure 8: *Time plots* z_n *corresponding to the initial conditions* $(0.4, 0.3, 0.2) \& (0.4, 0.3, 0.2001)$.

In addition, Figure (8) and (9) shows that sensitive dependence on initial conditions, $x \& z$ -coordinates of the four orbits, for system (2), is plotted against the time with the parameter constellation $r = 1.067$, $\tau = 1.39$, $\eta = 0.79$, $a = 2.89$, $\rho = 0.77$, $\nu = 0.75$, $\mu = 0.42$. The $x \& z$ -coordinates of initial conditions differ by *0.0001* and the other coordinates are kept equal.

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