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Existence of Periodic Orbits of the First kind in the CR4BP when the Second Primary is an Oblate Spheroid and Third Primary is a Triaxial Rigid body

Payal Singh[1], M. R. Hassan[2], Md. Aminul Hassan[3]

[1]*Asst. Teacher, Maths, P.G.I. School, Bhagalpur, India, satyamshivam7707@gmail.com*
[2]*Department of Mathematics, S.M. College, T.M.B. University, Bhagalpur, India, hassansmc@gmail.com*
[3]*GTE, Bangalore, India, mahassan012@gmail.com*

Abstract— We proposed to study the existence of periodic orbits of the first kind in the CR4BP when the second primary is an Oblate spheroid, the third primary is a triaxial rigid body and the fourth primary is of comparatively smaller mass placed at triangular libration point. By applying the model of Hassan [1,2] and Payal [3], we examine the existence of periodic orbits with the technique of Choudhary [4] and conditions of Duboshin [5] with different parametric values and found satisfied.

Keywords— CR4BP, Oblate Spheroid, Triaxial Rigid body, Regularization, Periodicity

I. INTRODUCTION

Giacaglia [6] applied the method of analytic continuation to examine the existence of periodic orbits of collision of the first kind in the Circular Restricted Four–body Problem (CR3BP). Bhatnagar [7] generalized the problem in elliptic case. Further Bhatnagar [8] extended the work of Giacaglia [6] in the Circular Restricted Four–body Problem (CR4BP) by considering three primaries at the vertices of an equilateral triangle. In last three decades a series of works have been performed by different authors with different perturbations in the circular and elliptic restricted three-body and four-body problem but nobody established the proper mathematical model of the Restricted Four-body Problem (R4BP).

Recently Ceccaroni and Biggs [9] studied the autonomous coplanar CR4BP with an extension to low-thrust propulsion for application to the future science mission. In their problem they also studied the stability region of the artificial and natural equilibrium points in the Sun-Jupiter Trojan Asteroid-Spacecraft system. Using the concept of Ceccaroni and Biggs [9] and the method of Hassan [1,2], we have proposed to study the existence of periodic orbits of the first kind in the autonomous restricted four–body problem (R4BP) by considering the second primary as an oblate spheroid and third primary as a triaxial rigid body.

II. EQUATIONS OF MOTION OF THE INFINITESIMAL MASS

Let $P_i\left(i=1,2,3\right)$ be the three primaries of masses $m_j\left(j=1,2,3\right)$ respectively, where $m_1 \geq m_2 > m_3$. The problem is the restricted four-body problem so the fourth body P of infinitesimal mass m is assumed to be so small that it can't influence the motion of the primaries but the motion of $P\left(m\right)$ is influenced by them. In addition, we assumed that the mass m_3 (mass of the third primary placed at L_4 of the R3BP) is small enough so that it can't influence the motion of the two dominating primaries P_1 and P_2 but can influence the motion of the infinitesimal body $P\left(m\right)$.

Thus, the centre of mass (i.e. the bary-centre) i.e. the centre of rotation of the system remains at the bary-centre O of the two primaries P_1 and P_2. Also, all the primaries P_1, P_2 and P_3 are moving in the same plane of motion in different circular orbits of radii OP_1, OP_2 and OP_3 respectively around the bary-centre O with the same angular velocity $\vec{\omega}$. Considering $\left(O, XY\right)$ as an inertial frame in such a way that the $XY-$plane coincides with the plane of motion of the primaries and origin coincides with O. Initially let the principal axes of the second primary P_2 are parallel to the synodic axes $\left(O, xy\right)$ and its axis of symmetry is perpendicular to the plane of motion. Since the primaries are revolving without rotation about O with the same angular velocity as that of the synodic axes hence, the principal axes of P_2 will remain parallel to the co-ordinate axes throughout the motion.

Let at any time $t, P_1\left(\xi_1, 0\right)$ and $P_2\left(\xi_2, 0\right)$ be the positions of two dominating primaries on the $x-$axis of the rotating (synodic) co-ordinate system and $P_3\left(\xi_3, \eta_3\right)$ be the third primary placed at the equilibrium point L_4 of P_1 and P_2. Let $\vec{r}_1, \vec{r}_2$ and $\vec{r}_3$ be the displacements of P_1, P_2 and P_3 relative to P and $\vec{r}$ be the position vector of $P\left(x, y\right)$, then

$$\left.\begin{aligned} &\vec{r}_1=\left(x-\xi_1\right)\hat{i}+y\hat{j}=\overrightarrow{P_1P}, && \vec{r}_2=\left(x-\xi_2\right)\hat{i}+y\hat{j}=\overrightarrow{P_2P}, \\ &\vec{r}_3=\left(x-\xi_3\right)\hat{i}+y\left(y-\eta_3\right)\hat{j}=\overrightarrow{P_3P}, && \vec{r}=x\hat{i}+y\hat{j}=\overrightarrow{OP}. \end{aligned}\right\} \tag{1}$$

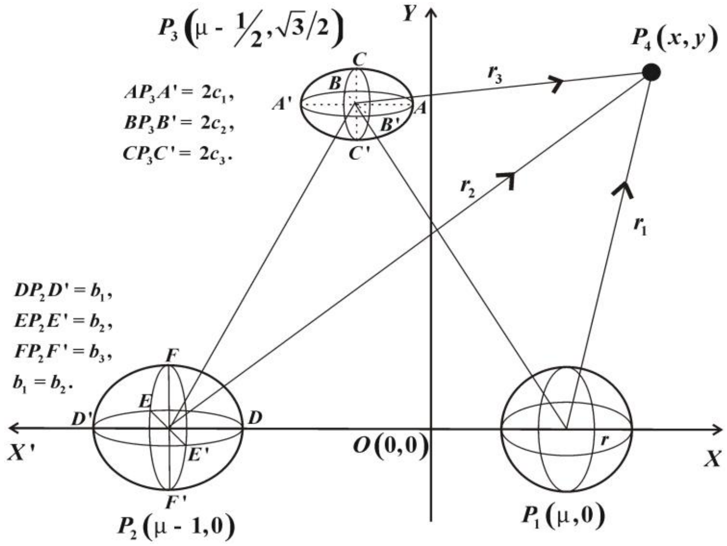

Fig. 1: Configuration of CR4BP when Second Primary is an Oblate Spheroid and Third Primary is a Triaxial Rigid body

Let $\vec{F}_1, \vec{F}_2$ and $\vec{F}_3$ be the gravitational forces exerted by the primaries P_1, P_2 and P_3 respectively on the infinitesimal mass m at $P(x,y)$, then

$$\vec{F}_1 = -\frac{Gmm_1}{r_1^3}\left\{(x-\xi_1)\hat{i} + y\hat{j}\right\} \tag{2}$$

Let b_1, b_2 and b_3 be the lengths of the semi-axes of the second primary $P_2(\xi_2, 0)$ then the gravitational force exerted by $P_2(\xi_2, 0)$ on $P(x,y)$ is given by McCuskey [10]

$$\vec{F}_2 = -\frac{Gmm_2}{r_2^3}\hat{r}_2 - \frac{3Gmm_2\sigma_1}{2r_2^4}\hat{r}_2$$

where $\sigma_1 = \dfrac{\left(b_1^2 - b_2^2\right)}{5R^2}$, R is the dimensional distances between the main primaries P_1 and P_2.

Here

$\hat{r}$ = unit vector along $\overrightarrow{P_2P}$ so $\hat{r} = \dfrac{\vec{r}_2}{|\vec{r}_2|} = \dfrac{(x-\xi_2)\hat{i} + y\hat{j}}{r_2}$,

$$\therefore \vec{F} = -Gmm_2\left[\left\{\frac{x-\xi_2}{r_2^3} + \frac{3(x-\xi_2)\sigma_1}{2r_2^5}\right\}\hat{i} + \left\{\frac{y}{r_2^3} + \frac{3y\sigma_1}{2r_2^5}\right\}\hat{j}\right] \tag{3}$$

Let c_1, c_2, c_3 be the semi-axes of the third primary at $P_3(\xi_3, \eta_3) \equiv L_4$ then gravitational force exerted by $P_3(\xi_3, \eta_3)$ on $P(x,y)$ is given by

$$\vec{F}_3 = -\frac{Gmm_3}{r_3^3}\hat{r}_3 - \frac{3Gmm_3}{2r_3^4}\left(\frac{2c_1^2 - c_2^2 - c_3^2}{5R^2}\right)\hat{r}_3 + \frac{15Gmm_3}{2r_3^6}\frac{c_1^2 - c_2^2}{5R^2}(y-\eta_3)^2\hat{r}_3 \text{ where } \hat{r}_3 = \frac{(x-\xi_3)\hat{i} + (y-\eta_3)\hat{j}}{r_3}.$$

Taking $\sigma_1' = \dfrac{c_1^2 - c_3^2}{5R^2},\ \sigma_2' = \dfrac{c_2^2 - c_3^2}{5R^2}$, (4)

then

$$\vec{F}_3 = -\frac{Gmm_3}{r_3^3}\hat{r}_3 - \frac{3Gmm_3}{2r_3^4}\left(2\sigma_1' - \sigma_2'\right)\hat{r}_3 + \frac{15Gmm_3}{2r_3^6}\left(\sigma_1' - \sigma_2'\right)\hat{r}_3(y-\eta_3)^2$$

Total gravitational force exerted by the three primaries on the infinitesimal mass at $P(x,y)$ is given by

$$\left.\begin{aligned}\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = -Gm\Bigg[\Bigg\{&\frac{m_1(x-\xi_1)}{r_1^3} + \frac{m_2(x-\xi_2)}{r_2^3} + \frac{m_3(x-\xi_3)}{r_3^3} + \frac{3m_2\sigma_1(x-\xi_2)}{2r_2^5} + \frac{3m_3\left(2\sigma_1' - \sigma_2'\right)(x-\xi_3)}{2r_3^5} \\ &-\frac{15m_2\left(\sigma_1' - \sigma_2'\right)(x-\xi_3)}{2r_2^7}(y-\eta_3)^2\Bigg\}\hat{i} + \left\{\frac{m_1 y}{r_1^3} + \frac{m_2 y}{r_2^3} + \frac{m_3(y-\eta_3)}{r_3^3} + \frac{3m_2\sigma_1}{2r_2^5}y + \frac{3m_3\left(2\sigma_1' - \sigma_2'\right)}{2r_3^5}(y-\eta_3)\right. \\ &\left.-\frac{15m_3\left(\sigma_1' - \sigma_2'\right)}{2r_3^7}(y-\eta_3)^3\right\}\hat{j}\Bigg].\end{aligned}\right\} \tag{5}$$

The equation of motion of the infinitesimal mass in the gravitational field of the three primaries P_1, P_2 and P_3 is given by

$$m\left[\frac{\partial^2\vec{r}}{\partial t^2}+2\vec{\omega}\times\frac{\partial\vec{r}}{\partial t}+\frac{\partial\vec{\omega}}{\partial t}\times\vec{r}+\vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right)\right]=\vec{F}, \tag{6}$$

where

$$\frac{\partial^2\vec{r}}{\partial t^2}=\ddot{x}\hat{i}+\ddot{y}\hat{j}=\text{relative acceleration},$$

$$\vec{\omega}\times\frac{\partial\vec{r}}{\partial t}=-n\dot{y}\hat{i}+n\dot{x}\hat{j}=\text{coriolis acceleration},$$

$$\text{Euler's acceleration}=\frac{\partial\vec{\omega}}{\partial t}\times\vec{r},\qquad\left(\text{as }\vec{\omega}=n\hat{k}\text{ is a constant vector}\right)$$

$$\vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right)=-n^2x\hat{i}-n^2y\hat{j}=\text{centrifugal acceleration}.$$

From Equations (5) and (6), we get

$$m\left[\left(\ddot{x}-2n\dot{y}-n^2x\right)\hat{i}+\left(\ddot{y}+2n\dot{x}-n^2y\right)\hat{j}\right]=-Gm\left[\left\{\frac{m_1\left(x-\xi_1\right)}{r_1^3}+\frac{m_2\left(x-\xi_2\right)}{r_2^3}+\frac{m_3\left(x-\xi_3\right)}{r_3^3}+\frac{3m_2\sigma_1\left(x-\xi_2\right)}{2r_2^5}\right.\right.$$
$$\left.+\frac{3m_3\left(2\sigma_1^{'}-\sigma_2^{'}\right)\left(x-\xi_3\right)}{2r_3^5}-\frac{15m_3\left(\sigma_1^{'}-\sigma_2^{'}\right)\left(x-\xi_3\right)}{2r_2^7}\left(y-\eta_3\right)^2\right\}\hat{i}+\left\{\frac{m_1y}{r_1^3}+\frac{m_2y}{r_2^3}+\frac{m_3\left(y-\eta_3\right)}{r_3^3}+\frac{3m_2\sigma_1}{2r_2^5}y\right.$$
$$\left.\left.+\frac{3m_3\left(2\sigma_1^{'}-\sigma_2^{'}\right)}{2r_3^5}\left(y-\eta_3\right)-\frac{15m_3\left(\sigma_1^{'}-\sigma_2^{'}\right)}{2r_3^7}\left(y-\eta_3\right)^3\right\}\hat{j}\right]$$

By equating the coefficients of $\hat{i}$ and $\hat{j}$ from both sides, we get the equations of motion of the infinitesimal mass as

$$\ddot{x}-2n\dot{y}-n^2x=-G\left[\frac{m_1\left(x-\xi_1\right)}{r_1^3}+\frac{m_2\left(x-\xi_2\right)}{r_2^3}+\frac{m_3\left(x-\xi_3\right)}{r_3^3}+\frac{3m_2\sigma_1\left(x-\xi_3\right)}{2r_2^5}+\frac{3m_3\left(2\sigma_1^{'}-\sigma_2^{'}\right)\left(x-\xi_3\right)}{2r_3^5}\right.$$
$$\left.-\frac{15m_3\left(\sigma_1^{'}-\sigma_2^{'}\right)\left(x-\xi_3\right)}{2r_3^7}\left(y-\eta_3\right)^2\right], \tag{7}$$

$$\ddot{y}+2n\dot{x}-n^2y=-G\left[\frac{m_1y}{r_1^3}+\frac{m_2y}{r_2^3}+\frac{m_3\left(y-\eta_3\right)}{r_3^3}+\frac{3m_2\sigma_1}{2r_2^5}y+\frac{3m_3\left(2\sigma_1^{'}-\sigma_2^{'}\right)}{2r_3^5}\left(y-\eta_3\right)-\frac{15m_3\left(\sigma_1^{'}-\sigma_2^{'}\right)}{2r_3^7}\left(y-\eta_3\right)^3\right] \tag{8}$$

Let $\vec{v}=v_1\hat{i}+v_2\hat{j}$ be the linear velocity of the infinitesimal mass at $P\left(x,y\right)$ then

$$\vec{v}=\frac{d\vec{r}}{dt}=\frac{\partial\vec{r}}{\partial t}+\vec{\omega}\times\vec{r}=\left(\dot{x}-ny\right)\hat{i}+\left(\dot{y}+nx\right)\hat{j}=v_1\hat{i}+v_2\hat{j},\qquad\left[\text{as }\frac{d}{dt}=\frac{\partial}{\partial t}+\vec{\omega}\times\right]$$

where $v_1=\dot{x}-ny, v_2=\dot{y}+nx$

$\therefore$ Kinetic energy of the infinitesimal mass is given by

$$T=\frac{1}{2}\left|\vec{v}\right|^2=\frac{1}{2}\left(\dot{x}^2+\dot{y}^2\right)+n\left(x\dot{y}-\dot{x}y\right)+\frac{n^2}{2}\left(x^2+y^2\right)\text{ for unit mass of the infinitesimal body.} \tag{9}$$

where the mean motion of the synodic frame is given by

$$n^2=1+\frac{3}{2}\sigma_1+\frac{3}{2}\left(2\sigma_1^{'}-\sigma_2^{'}\right). \tag{10}$$

Let p_1 and p_2 be the momenta corresponding to the co-ordinates x and y respectively then $p_1=\frac{\partial T}{\partial\dot{x}}, p_2=\frac{\partial T}{\partial\dot{y}}$

$\Rightarrow p_1=\dot{x}-ny=v_1$ and $p_2=\dot{y}+nx=v_2$

$$\text{Thus } T=\frac{1}{2}\left(p_1^2+p_2^2\right) \tag{11}$$

Let $V_i=\left(i=1,2,3\right)$ be the gravitational potential of the primaries of masses $m_i\left(i=1,2,3\right)$ at any point outside of $P\left(x,y\right)$, then

$$\left.\begin{aligned}&V_1=-\frac{Gm_1}{r_1},\qquad V_2=-\frac{Gm_2}{r_2}-\frac{Gm_2\sigma_1}{2r_2^3},\\&V_3=-\frac{Gm_3}{r_3}-\frac{Gm_3\left(2\sigma_1^{'}-\sigma_2^{'}\right)}{2r_3^3}+\frac{3Gm_3\left(\sigma_1^{'}-\sigma_2^{'}\right)}{2r_3^5}\left(y-\eta_3\right)^2.\end{aligned}\right\} \tag{12}$$

$\therefore$ Total potential at any point outside of $P(x,y)$ due to three primaries is given by

$$V=\sum_{i=1}^{3}V_i=-G\left(\frac{m_1}{r_1}+\frac{m_2}{r_2}+\frac{m_3}{r_3}\right)-\frac{Gm_2\sigma_1}{2r_2^3}+\frac{3Gm_3\left(\sigma_1^{'}-\sigma_2^{'}\right)}{2r_3^5}\left(y-\eta_3\right)^2-\frac{Gm_3\left(2\sigma_1^{'}-\sigma_2^{'}\right)}{2r_3^3}. \tag{13}$$

The Lagrangian of the infinitesimal mass is given by

$$L=T-V=\frac{1}{2}\left(p_1^2+p_2^2\right)+G\left(\frac{m_1}{r_1}+\frac{m_2}{r_2}+\frac{m_3}{r_3}\right)+\frac{Gm_2\sigma_1}{2r_2^3}+\frac{Gm_3\left(2\sigma_1^{'}-\sigma_2^{'}\right)}{2r_3^3}-\frac{3Gm_3\left(\sigma_1^{'}-\sigma_2^{'}\right)}{2r_3^5}\left(y-\eta_3\right)^2. \tag{14}$$

The Hamiltonian of the infinitesimal body of unit mass is given by

$$H=\sum p\dot{x}-L=\left(p_1\dot{x}+p_2\dot{y}\right)-L$$

$$H=\frac{1}{2}\left(p_1^2+p_2^2\right)+n\left(p_1y-p_2x\right)-G\left(\frac{m_1}{r_1}+\frac{m_2}{r_2}+\frac{m_3}{r_3}\right)-\frac{Gm_2\sigma_1}{2r_2^3}-\frac{Gm_3\left(2\sigma_1^{'}-\sigma_2^{'}\right)}{2r_3^3}+\frac{3Gm_3\left(\sigma_1^{'}-\sigma_2^{'}\right)}{2r_3^5}\left(y-\eta_3\right)^2 \tag{15}$$

$=C=$ constant.

Assuming μ as the mass ratio of m_2 and ε as the mass ratio of m_3 to the total mass of the dominating primaries P_1 and P_2 then $\mu=\dfrac{m_2}{m_1+m_2}$ and $\varepsilon=\dfrac{m_3}{m_1+m_2}$. Also assuming $m_1+m_2=1$ then $m_2=\mu, m_1=1-\mu$ and $m_3=\varepsilon$. From the definition of the centre of mass of m_1 and m_2, we have $m_1\xi_1+m_2\xi_2=0$ which implies $\xi_1=\mu,\xi_2=\mu-1$, $\xi_3=\mu-\dfrac{1}{2}$ and $\eta_3=\dfrac{\sqrt{3}}{2}$. Thus the co-ordinates of the three primaries P_1, P_2 and P_3 are $(\mu,0),(\mu-1,0)$ and $\left(\mu-\dfrac{1}{2},\dfrac{\sqrt{3}}{2}\right)$ respectively, which confirms $\left|\overrightarrow{P_1P_2}\right|=\left|\overrightarrow{P_2P_3}\right|=\left|\overrightarrow{P_3P_1}\right|=1$ i.e. $P_1P_2P_3$ is an equilateral triangle of sides of unit length.

Now choosing unit of time in such a way that $G=1$ and taking $x=x_1$ and $y=x_2$, then the reduced Hamiltonian is given by

$$H=\frac{1}{2}\left(p_1^2+p_2^2\right)+n\left(p_1x_2-p_2x_1\right)-\frac{1-\mu}{r_1}-\frac{\mu}{r_2}-\frac{\varepsilon}{r_3}-\frac{\mu\sigma_1}{2r_2^3}+\frac{3\varepsilon\left(\sigma_1^{'}-\sigma_2^{'}\right)}{2r_3^5}\left(y-\eta_3\right)^2=C=\text{constant}. \tag{16}$$

The Hamiltonian – Canonical equations are

$$\frac{dx_i}{dt}=\frac{\partial H}{\partial p_i},\qquad \frac{dp_i}{dt}=-\frac{\partial H}{\partial x_i}.\quad (i=1,2) \tag{17}$$

The energy integral of the infinitesimal mass is

$$\frac{1}{2}\left(\dot{x}^2+\dot{y}^2\right)=\frac{1}{2}n^2\left(x^2+y^2\right)+\frac{1-\mu}{r_1}+\frac{\mu}{r_2}+\frac{\varepsilon}{r_3}+\frac{\mu\sigma_1}{2r_2^3}+\frac{\varepsilon\left(2\sigma_1^{'}-\sigma_2^{'}\right)}{2r_3^3}-\frac{3\varepsilon\left(\sigma_1^{'}-\sigma_2^{'}\right)}{2r_3^5}\left(y-\eta_3\right)^2. \tag{18}$$

III. REGULARIZATION

In our Hamiltonian given in Equation (16), there are three singularities $r_1=r_2=r_3=0$, so to examine the existence of periodic orbits around the first primary, we have to eliminate the singularity $r_1=0$ from the Hamiltonian in Equation (16). For this, let us define an extended generating function S by

$$S=\left(\mu+q_1^2-q_2^2\right)p_1+2q_1q_2p_2, \tag{19}$$

where $Q_i\,(i=1,2)$ are momenta associated with new co-ordinates $q_i\,(i=1,2)$ and $x_i=\dfrac{\partial S}{\partial p_i}, Q_i=\dfrac{\partial S}{\partial q_i}$.

Clearly, $x_1=\dfrac{\partial S}{\partial p_1}=\mu+q_1^2-q_2^2$ and $x_2=\dfrac{\partial S}{\partial p_2}=2q_1q_2.$ (20)

$$Q_1=2\left(p_1q_1+p_2q_2\right)\quad\text{and}\quad Q_2=2\left(p_2q_1-p_1q_2\right). \tag{21}$$

$$r_1^2=\left(x_1-\mu\right)^2+x_2^2=\left(q_1^2-q_2^2\right)^2+4q_1^2q_2^2=\left(q_1^2+q_2^2\right)^2$$

$$r_1=q_1^2+q_2^2,\qquad r_2^2=1+r_1^2+2\left(q_1^2-q_2^2\right),\qquad r_3^2=1+r_1^2+\left(q_1^2-q_2^2\right)-2\sqrt{3}q_1q_2. \tag{22}$$

From Equation (21), we have

$$p_1=\frac{1}{2r_1}\left(Q_1q_1-Q_2q_2\right)\quad\text{and}\quad p_2=\frac{1}{2r_1}\left(Q_1q_2+Q_2q_1\right). \tag{23}$$

$$\therefore\ p_1^2+p_2^2=\frac{1}{4r_1}\left(Q_1^2+Q_2^2\right). \tag{24}$$

$$n\left(p_1x_2-p_2x_1\right)=\frac{n}{2}\left(Q_1q_2-Q_2q_1\right)-\frac{n\mu}{2r_1}\left(Q_1q_2+Q_2q_1\right). \tag{25}$$

The combination of Equations (15), (24) & (25) gives the Hamiltonian H in terms of new variables $q_i, Q_i \left(i=1,2\right)$ as

$$\begin{aligned} H &= \frac{1}{8r_1}\left(Q_1^2+Q_2^2\right)+\frac{1}{2}n\left(Q_1q_2-Q_2q_1\right)-\frac{n\mu}{2r_1}\left(Q_1q_2+Q_2q_1\right)-\frac{1-\mu}{r_1}-\frac{\mu}{r_2}-\frac{\varepsilon}{r_3}-\frac{\mu\sigma_1}{2r_2^3}-\frac{\varepsilon\left(2\sigma_1^{'}-\sigma_2^{'}\right)}{2r_3^5} \\ &+\frac{3\varepsilon\left(\sigma_1^{'}-\sigma_2^{'}\right)}{2r_3^5}\left(2q_1q_2-\frac{\sqrt{3}}{2}\right)^2=C. \end{aligned} \tag{26}$$

Let us introduce pseudo time τ by the equation

$$dt = r_1 d\tau \qquad \left(\tau = 0 \text{ when } t = 0\right). \tag{27}$$

The Canonical equations of motion corresponding to the regularized Hamiltonian K are given by

$$\frac{dq_i}{d\tau}=\frac{\partial K}{\partial Q_i} \quad \text{and} \quad \frac{dQ_i}{d\tau}=-\frac{\partial K}{\partial q_i} \quad \left(i=1,2\right) \tag{28}$$

where the regularized Hamiltonian K is given by

$$\left.\begin{aligned} K &= r_1\left(H-C\right)=0, \\ &=\frac{1}{8}\left(Q_1^2+Q_2^2\right)+\frac{1}{2}nr_1\left(Q_1q_2-Q_2q_1\right)-\frac{n\mu}{2}\left(Q_1q_2+Q_2q_1\right)-\left(1-\mu\right)-\frac{\mu r_1}{r_2}-\frac{\varepsilon r_1}{r_3}-\frac{\mu r_1\sigma_1}{2r_2^3}-\frac{\varepsilon r_1\left(2\sigma_1^{'}-\sigma_2^{'}\right)}{2r_3^3} \\ &+\frac{3\varepsilon r_1\left(\sigma_1^{'}-\sigma_2^{'}\right)}{2r_3^5}\left(2q_1q_2-\frac{\sqrt{3}}{2}\right)-r_1C=0. \end{aligned}\right\} \tag{29}$$

Since ε is very-very small in comparison of the masses of the dominating primaries hence $\forall\varepsilon \in \left]0,\mu\right[$, we can take $\varepsilon = \mu\varepsilon_0$ and $C = C_0+\mu C_1+\mu^2C_2+\mu^3C_3+.....$. Let us write $K = K_0+\mu K_1 = 0$ then from Equation (29), we have

$$K_0 = \frac{1}{8}\left(Q_1^2+Q_2^2\right)+\frac{1}{2}r_1\left[n\left(Q_1q_2-Q_2q_1\right)-2C_0\right]-1=-\lambda\left(\text{say}\right), \tag{30}$$

$$\left.\begin{aligned} K_1 &= 1-\frac{n}{2}\left(Q_1q_2+Q_2q_1\right)-r_1\left[C_1+\frac{1}{r_2}+\frac{\varepsilon_0}{r_3}+\frac{\sigma_1}{2r_2^3}+\frac{A^{'}}{r_3^3}-\frac{B^{'}}{r_3^5}\left(2q_1q_2-\frac{\sqrt{3}}{2}\right)^2\right], \\ &= 1-\frac{n}{2}\left(Q_1q_2+Q_2q_1\right)-r_1\left[c_1+\frac{1}{r_2}+\frac{\varepsilon_0}{r_3}+\frac{\sigma_1}{2r_2^3}+\frac{A^{'}}{r_3^3}-\frac{4B^{'}}{r_3^5}q_1^2q_2^2-\frac{\sqrt{3}B^{'}}{r_3^5}q_1q_2+\frac{3B^{'}}{4r_3^5}\right], \end{aligned}\right\} \tag{31}$$

where $A' = \frac{\varepsilon}{2}\left(2\sigma_1^{'}-\sigma_2^{'}\right), B' = \frac{3}{2}\left(\sigma_1-\sigma_2\right)\varepsilon_0$.

IV. GENERATING SOLUTION

For generating solution, we shall choose K_0 for our Hamiltonian function, so in order to solve the Hamilton – Jacobi equation associated with K_0, let us write $Q_i = \frac{\partial W}{\partial q_i}\left(i=1,2\right)$ and $1-\lambda = \alpha > 0$ arbitrary constant. Since t is not involved explicitly in K_0 hence the Hamilton – Jacobi equation may be written as

$$\frac{1}{8}\left[\left(\frac{\partial W}{\partial q_1}\right)^2+\left(\frac{\partial W}{\partial q_2}\right)^2\right]+\frac{1}{2}r_1\left[n\left(q_2\frac{\partial W}{\partial q_1}-q_1\frac{\partial W}{\partial q_2}\right)-2C_0\right]=\alpha. \tag{32}$$

$$\left.\begin{aligned} &\text{Putting } q_1 = \rho\cos\varphi,\ q_2 = \rho\sin\varphi \\ &\text{then } \rho^2 = q_1^2+q_2^2 = r_1 \text{ and } \varphi = \tan^{-1}\left(\frac{q_2}{q_1}\right) \end{aligned}\right\} \tag{33}$$

Now $W = W\left(q_1,q_2\right) = W\left(\rho,\varphi\right)$

$$\Rightarrow Q_1 = \frac{\partial W}{\partial q_1}=\frac{\partial W}{\partial \rho}\cos\varphi-\frac{\partial W}{\partial \varphi}.\frac{\sin\varphi}{\rho} \quad \text{and} \quad Q_2 = \frac{\partial W}{\partial q_2}=\frac{\partial W}{\partial \rho}\sin\varphi+\frac{\partial W}{\partial \varphi}.\frac{\cos\varphi}{\rho} \tag{34}$$

$$\therefore \left(\frac{\partial W}{\partial q_1}\right)^2+\left(\frac{\partial W}{\partial q_2}\right)^2=\left(\frac{\partial W}{\partial \rho}\right)^2+\frac{1}{\rho^2}\left(\frac{\partial W}{\partial \varphi}\right)^2 \text{ and } q_2\frac{\partial W}{\partial q_1}-q_1\frac{\partial W}{\partial q_2}=-\frac{\partial W}{\partial \varphi}.$$

Thus, the Equation (32) reduces to

$$\frac{1}{8}\left[\left(\frac{\partial W}{\partial \rho}\right)^2+\frac{1}{\rho^2}\left(\frac{\partial W}{\partial \varphi}\right)^2\right]+\frac{1}{2}\rho^2\left[-n\frac{\partial W}{\partial \varphi}-2C_0\right]=\alpha. \tag{35}$$

This is a partial differential equation of second degree, so by the method of variable separable, the solution of Equation (35) may be written as

$$W = U(\rho) + 2G\phi, \tag{36}$$

where G is an arbitrary constant.

Now introducing a new variable z by $r_1 = \rho^2 = z$ then $\frac{dz}{d\rho} = 2\rho$

$$\therefore \frac{\partial W}{\partial \rho} = \frac{\partial U}{\partial \rho} = \frac{dU}{d\rho} = \frac{dU}{dz}.\frac{dz}{d\rho} = 2\rho\frac{dU}{dz}$$

$$\text{i.e., } \frac{\partial W}{\partial \rho} = 2\rho\frac{dU}{dz} \quad \text{and} \quad \frac{dW}{d\varphi} = 2G. \tag{37}$$

Introducing Equation (37) in Equation (36), we get

$$\frac{1}{8}\left[\left(2\rho\frac{dU}{dz}\right)^2 + \frac{1}{\rho^2}(2G)^2\right] + \frac{1}{2}\rho^2\left[-n.2G - 2C_0\right] = \alpha,$$

$$\left(\frac{dU}{dz}\right)^2 = -\frac{2(nG + C_0)}{z^2}F(z),$$

where $F(z) = -z^2 - \frac{\alpha z}{nG + C_0} + \frac{G^2}{2(nG + C_0)}$ is a quadratic expression in z. (38)

$$\text{Thus } \frac{dU}{dz} = \sqrt{-2(nG + C_0)}\frac{\sqrt{F(z)}}{z} \tag{39}$$

$$U(z, \varsigma, \alpha) = \sqrt{-2(nG + C_0)}\int_{z_1}^{z}\frac{\sqrt{F(z)}}{z}dz \tag{40}$$

where z_1 is the smaller root of the equation $F(z) = 0$.

From Equation (40) we conclude that for general solution, we need only two arbitrary constants assigned as α and G. Therefore, the solution (40) may be regarded as a general solution. Following Giacaglia [6] and Bhatnagar [8], let us introduce the parameters n, a, e, l by the relations

$$z_1 = na(1 - e), \quad z_2 = an(1 + e) \quad \text{and} \quad z = z_1\cos^2\frac{l}{2} + z_2\sin^2\frac{l}{2} = na(1 - e\cos l). \tag{41}$$

where z_1 and z_2 are the other roots of the equation $F(z) = 0$, a is the semi-major axis, e is the eccentricity and l is the semi-latus rectum of the elliptic orbit of the infinitesimal mass around the first primary. It may be noted that for $z = z_1, l = 0$.

From Equation (41),

$$z_1 + z_2 = 2na, \quad z_1 z_2 = n^2a^2\left(1 - e^2\right), \tag{42}$$

Since z_1 and z_2 are the roots of the equation $F(z) = 0$ hence from Equation (38),

$$\text{i.e., } z^2 + \frac{\alpha z}{nG + C_0} - \frac{G^2}{2(nG + C_0)} = 0,$$

$$z_1 + z_2 = -\frac{\alpha}{nG + C_0} \quad \text{and} \quad z_1 z_2 = -\frac{G^2}{2(nG + C_0)}. \tag{43}$$

From Equations (42) and (43), we have

$$2na = -\frac{\alpha}{nG + C_0},$$

$$n^2a^2\left(1 - e^2\right) = -\frac{G^2}{2(nG + C_0)},$$

$$\Rightarrow a = -\frac{\alpha}{2n(nG + C_0)} = \frac{\alpha}{n\left[-2(nG + C_0)\right]}.$$

Introducing a new parameter L by the relation

$$\alpha = \bar{L}\left[-2(nG + C_0)\right]^{\frac{1}{2}} > 0, \tag{44}$$

$$\text{then } a = \frac{\bar{L}}{n\left[-2(nG + C_0)\right]^{\frac{1}{2}}} > 0. \tag{45}$$

$$\text{Also } n^2a^2\left(1 - e^2\right) = -\frac{G^2}{2(nG + C_0)},$$

$$n^2\left(1-e^2\right)\frac{\bar{L}^2}{n^2\left[-2\left(nG+C_0\right)\right]}=\frac{G^2}{\left[-2\left(nG+C_0\right)\right]},$$

$$\Rightarrow e^2=\left(1-\frac{G^2}{\bar{L}^2}\right)^{1/2}\leq 1. \tag{46}$$

From Equation (38),

$$F(z)=-z^2-\frac{\alpha z}{\left(nG+C_0\right)}+\frac{G^2}{2\left(nG+C_0\right)},$$
$$=n^2a^2e^2-n^2a^2e^2\cos^2 l,$$
$$F(z)=n^2a^2e^2\sin^2 l. \tag{47}$$

The Hamilton-Canonical equation of motion corresponding to the Hamiltonian K_0 are given by

$$\left.\begin{array}{ll}\dfrac{dq_1}{d\tau}=\dfrac{\partial K_0}{\partial Q_1}, & \dfrac{dq_2}{d\tau}=\dfrac{\partial K_0}{\partial Q_2},\\[2ex] \dfrac{dQ_1}{d\tau}=-\dfrac{\partial K_0}{\partial q_1}, & \dfrac{dQ_2}{d\tau}=-\dfrac{\partial K_0}{\partial q_2},\end{array}\right\} \tag{48}$$

where $K_0=\frac{1}{8}\left(Q_1^2+Q_2^2\right)+\frac{1}{2}\rho^2\left[n\left(Q_1q_2-Q_2q_1\right)-2C_0\right]-1.$

$$\Rightarrow\frac{\partial K_0}{\partial Q_1}=\frac{1}{4}Q_1+\frac{1}{2}\rho^2nq_2 \quad\text{and}\quad \frac{\partial K_0}{\partial Q_2}=\frac{1}{4}Q_2-\frac{1}{2}\rho^2nq_1.$$

Thus $q_1^{'}=\frac{1}{4}Q_1+\frac{1}{2}\rho^2nq_2$ and $q_2^{'}=\frac{1}{4}Q_2-\frac{1}{2}\rho^2nq_1$ (49)

where $\left(^{'}\right)$ denote the differentiation with respect to τ.

Now $\rho^2=q_1^2+q_2^2=z,$

$$\Rightarrow 2\rho\frac{d\rho}{dz}=2q_1\frac{dq_1}{d\tau}+2q_2\frac{dq_2}{d\tau}=\frac{dz}{d\tau},$$

$$\Rightarrow 2\rho\rho^{'}=2\left(q_1q_1^{'}+q_2q_2^{'}\right)=\frac{dz}{d\tau}. \tag{50}$$

But $q_1q_1^{'}+q_2q_2^{'}=q_1\left(\frac{1}{4}Q_1+\frac{1}{2}\rho^2nq_2\right)+q_2\left(\frac{1}{4}Q_2-\frac{1}{2}\rho^2nq_1\right)=\frac{1}{4}\left(q_1Q_1+q_2Q_2\right),$ [using Equation (49)]

$$\text{Thus } 2\rho\rho^{'}=2\sum_{i=1}^{2}q_iq_i^{'}=\frac{1}{2}\sum_{i=1}^{2}q_iQ_i=\frac{dz}{d\tau}. \tag{51}$$

Also $\sum_{i=1}^{2}q_iQ_i=q_1Q_1+q_2Q_2=q_1\left(\frac{\partial W}{\partial q_1}\right)+q_2\left(\frac{\partial W}{\partial q_2}\right),$

$$=\rho\cos\varphi\left(\cos\varphi\frac{\partial W}{\partial\rho}-\frac{\sin\varphi}{\rho}\frac{\partial W}{\partial\varphi}\right)+\rho\sin\varphi\left(\sin\varphi\frac{\partial W}{\partial\rho}+\frac{\cos\varphi}{\rho}\frac{\partial W}{\partial\varphi}\right),$$

$$=\rho\frac{\partial W}{\partial\rho}=\rho 2\rho\frac{dU}{dz}, \qquad \text{[using Equation (34)]}$$

$$=2\rho^2\frac{dU}{dz}=2z\frac{dU}{dz}, \qquad \text{[using Equation (37)]}$$

$$\Rightarrow\sum_{i=1}^{2}q_iQ_i=2z\left(\frac{dU}{dz}\right). \tag{52}$$

Also, from Equations (39), (51) and (52)

$$\frac{1}{2}\rho\frac{\partial W}{\partial\rho}=z\rho\rho^{'}=2\sum_{i=1}^{2}q_iq_i^{'}=\frac{1}{2}\sum_{i=1}^{2}q_iQ_i=z\frac{dU}{dz}=\sqrt{-2\left(nG+C_0\right)F(z)}=\frac{dz}{d\tau}. \tag{53}$$

From the last relation of Equation (53), we have

$$\Rightarrow\frac{dz}{d\tau}=\sqrt{-2\left(nG+C_0\right)}\sqrt{F(z)},$$

$$\Rightarrow\frac{dz}{\sqrt{F(z)}}=\sqrt{-2\left(nG+C_0\right)}d\tau,$$

$$\Rightarrow \int_{z_1}^{z} \frac{dz}{\sqrt{F(z)}} = \sqrt{-2(nG+C_0)} \int_{\tau_0}^{\tau} d\tau, \quad \text{where } \tau = z_1 \Rightarrow l = 0, z = \tau_0$$

$$\Rightarrow \int_0^l \frac{nal \sin l dl}{nal \sin l} = \sqrt{-2(nG+C_0)}(\tau - \tau_0), \qquad [\text{using Equations (41) and (48)}]$$

$$\Rightarrow l = \left[-2(nG+C_0)\right]^{\frac{1}{2}}(\tau - \tau_0),$$

$$\Rightarrow l = \int_{z_1}^{z} \frac{dz}{\sqrt{F(z)}} = \left[-2(nG+C_0)\right]^{\frac{1}{2}}(\tau - \tau_0). \tag{54}$$

Again, from Equation (53)

$$\frac{dz}{dt}.\frac{dt}{d\tau} = \sqrt{-2(nG+C_0)}\sqrt{F(z)},$$

$$\frac{dz}{dt} r_1 = \sqrt{-2(nG+C_0)}\sqrt{F(z)},$$

$$\Rightarrow z\frac{dz}{dt} = \sqrt{-2(nG+C_0)}\sqrt{F(z)},$$

$$\Rightarrow dt = \frac{1}{\left[-2(nG+C_0)\right]^{\frac{1}{2}}}.\frac{zdz}{\sqrt{F(z)}},$$

$$\Rightarrow \int_{t_0}^{t} dt = \frac{1}{\left[-2(nG+C_0)\right]^{\frac{1}{2}}} \int_0^l \frac{an(1-e\cos l) ane \sin l dl}{ane \sin l},$$

$$t - t_0 = \frac{an}{\left[-2(n\varsigma + C_0)\right]^{\frac{1}{2}}}(l - e\sin l) \qquad \text{where } t_0 \text{ is a constant.} \tag{55}$$

Now taking $\bar{L}$ and $\bar{G}$ as arbitrary constants in line of α and $\bar{G}$ and the solutions may be given by the relations

$$\frac{\partial W}{\partial \bar{L}} = \frac{\partial U}{\partial \bar{L}} = \int_{z_1}^{z} \frac{dz}{\sqrt{F(z)}} = l \quad \text{and} \quad \frac{\partial W}{\partial G} = \frac{\partial U}{\partial G} + 2\varphi = g. \tag{56}$$

From Equation (40),

$$U(z,G,L) = \left[-2(nG+C_0)\right]^{\frac{1}{2}} \int_{z_1}^{z} \sqrt{F(z)}\frac{dz}{z}.$$

Differentiating partially with respect to G, we get

$$\frac{\partial U}{\partial G} = \frac{\partial}{\partial G}\int_{z_1}^{z} \sqrt{-2(nG+C_0)F(z)}\frac{dz}{z},$$

$$= \int_{z_1}^{z} \frac{\partial}{\partial \bar{G}} \sqrt{-2(n\bar{G}+C_0)F(z)}\frac{dz}{z},$$

$$= \frac{n\sqrt{\bar{L}^2 - \bar{G}^2}\sin l}{2(n\bar{G}+C_0)} - \sqrt{1-e^2}\int_0^l \frac{dl}{(1-e\cos l)},$$

$$\Rightarrow \frac{\partial U}{\partial \bar{G}} = \frac{n\sqrt{\bar{L}^2 - \bar{G}^2}\sin l}{2(n\bar{G}+C_0)} - f,$$

where $f = \sqrt{1-e^2}\int_0^l \frac{dl}{(1-e\cos l)} \qquad (e \neq 1).$ (57)

From Equation (56),

$$g = \frac{\partial U}{\partial G} + 2\varphi,$$

$$\Rightarrow g = 2\varphi + \frac{n\sqrt{\bar{L}^2 - \bar{G}^2 \sin l}}{2(n\bar{G}+C_0)} - f, \tag{58}$$

$$\Rightarrow \varphi = \frac{1}{2}(g+f) - \frac{n\sqrt{L^2 - \bar{G}^2}}{4(n\bar{G}+C_0)}\sin l \quad \text{and} \quad \varphi = \frac{1}{2}g - \frac{nL}{4C_0}\sin l \tag{59}$$

where $(e \neq 1, G \neq 0, f \neq 0), (e = 1, \bar{G} = 0, f = 0)$.

Now let us find the value of K_0 in terms of $l, g, \bar{L}, \bar{G}$. For this, we have

$$K_0 = \frac{1}{8}\left(Q_1^2 + Q_2^2\right) + \frac{1}{2}\rho^2\left[n\left(Q_1 q_2 - Q_2 q_1\right) - 2C_0\right] - 1,$$

$$= \frac{1}{8}\left[\left(\frac{\partial W}{\partial \rho}\right)^2 + \left(\frac{\partial W}{\partial \varphi}\right)^2\right] + \frac{1}{2}\rho^2\left[-n\frac{\partial W}{\partial \varphi} - 2C_0\right] - 1,$$

$$\Rightarrow K_0 = L\left[-2\left(nG + C_0\right)\right]^{\frac{1}{2}}. \tag{60}$$

Therefore, for the problem generated by the Hamiltonian K_0, the equations of motion are

$$\left.\begin{aligned}
&\frac{dL}{d\tau} = \frac{\partial K_0}{\partial l} = 0 \Rightarrow L = \text{constant} = L_0, \qquad \frac{dG}{d\tau} = \frac{\partial K_0}{\partial g} = 0 \Rightarrow G = \text{constant} = \varsigma_0,\\
&\frac{dl}{d\tau} = -\frac{\partial K_0}{\partial L} = \left[-2\left(nG + C_0\right)\right]^{\frac{1}{2}} = \eta_l\,(\text{say}) \qquad \Rightarrow l = \eta_l \tau + l_0,\\
&\frac{dg}{d\tau} = -\frac{\partial K_0}{\partial G} = -\frac{L}{\left[-2\left(nG + C_0\right)\right]^{\frac{1}{2}}} = \eta_g\,(\text{say}) \quad \Rightarrow g = \eta_g \tau + g_0.
\end{aligned}\right\} \tag{61}$$

Further we are to express q_i and Q_i $(i = 1, 2)$ in terms of canonical elements $l, g, \bar{L}, \bar{G}$. From Equation (34),

$$Q_1 = \frac{\partial W}{\partial q_i} = \cos\varphi\frac{\partial W}{\partial \rho} - \frac{\sin\varphi}{\rho}\frac{\partial W}{\partial \varphi} = \cos\varphi 2\rho\frac{dU}{dz} - \frac{\sin\varphi}{\rho}\frac{\partial W}{\partial \varphi} = \frac{1}{\rho}\left[\cos\varphi 2\rho^2\frac{dU}{dz} - \sin\varphi 2G\right] = \frac{2}{\rho}\left[\cos\varphi z\frac{dU}{dz} - G\sin\varphi\right],$$

$$= \pm\frac{2}{\sqrt{z}}\left[\left\{-2\left(nG + C_0\right)\right\}^{\frac{1}{2}}\sqrt{F(z)}.\cos\varphi - G\sin\varphi\right] = \pm\frac{2}{\sqrt{z}}\left[na\left\{-2\left(nG + C_0\right)\right\}^{\frac{1}{2}} e\sin l\cos\varphi - G\sin\varphi\right],$$

i.e., $Q_1 = 2\left[\dfrac{eL\sin l\cos\varphi - G\sin\varphi}{\pm\sqrt{na\left(1 - e\cos l\right)}}\right].$

Thus,

$$\left.\begin{aligned}
&Q_1 = \pm\frac{2\left[eL\sin l\cos\varphi - G\sin\varphi\right]}{\sqrt{na\left(1 - e\cos l\right)}}, \qquad Q_2 = \pm\frac{2\left[eL\sin l\cos\varphi + G\sin\varphi\right]}{\sqrt{na\left(1 - e\cos l\right)}},\\
&q_1 = \pm\left[na\left(1 - e\cos l\right)\right]^{\frac{1}{2}}\cos\varphi, \qquad q_2 = \pm\left[na\left(1 - e\cos l\right)\right]^{\frac{1}{2}}\sin\varphi,
\end{aligned}\right\} \tag{62}$$

where φ is given by the first equation of (59). Where $e = 1, G = 0, f = 0,$ then the variables $q_i, Q_i\,(i = 1, 2)$ can be expressed in terms of canonical elements (l, g, L, G) as

$$\left.\begin{aligned}
&q_1 = \pm\sqrt{2an}\sin\frac{l}{2}\cos\varphi, \qquad q_2 = \pm\sqrt{2an}\sin\frac{l}{2}\sin\varphi,\\
&Q_1 = \pm\frac{4L}{\sqrt{2an}}\cos\frac{l}{2}\cos\varphi, \qquad Q_2 = \pm\frac{4L}{\sqrt{2an}}\cos\frac{l}{2}\sin\varphi,
\end{aligned}\right\} \tag{63}$$

where φ is given by the second equation of (59).

The original synodic cartesian co-ordinates in a uniformly rotating (synodic) system are obtained from the Equations (20) and (23) when $\mu = 0$,

$$\left.\begin{aligned}
&x_1 = q_1^2 - q_2^2, \qquad x_2 = 2q_1 q_2,\\
&p_1 = \frac{1}{2z}\left(Q_1 q_1 - Q_2 q_2\right), \qquad p_2 = \frac{1}{2z}\left(Q_2 q_1 - Q_1 q_2\right)
\end{aligned}\right\} \tag{64}$$

The sidereal cartesian co-ordinates are obtained by considering the transformation

$$\left.\begin{aligned}
&X_1 = x_1\cos nt - x_2\sin nt, \qquad X_2 = x_1\sin nt + x_2\cos nt,\\
&\dot{X}_1 = p_1\cos nt - p_2\sin nt, \qquad \dot{X}_2 = p_1\sin nt + p_2\cos nt,
\end{aligned}\right\} \tag{65}$$

where t is given by the Equation (55).

Now let us express K_1 in terms of the canonical elements $l, g, \bar{L}, \bar{G}$. From Equation (31),

$$K_1 = 1 - \frac{n}{2}\left(Q_1 q_2 + Q_2 q_1\right) - r_1\left[C_1 + \frac{1}{r_2} + \frac{\varepsilon_0}{r_3} + \frac{\sigma_1}{2r_2^3} + \frac{A'}{r_3^3} - \frac{4B' q_1^2 q_2^2}{r_3^5} + \frac{\sqrt{3}B' q_1 q_2}{r_3^5} + \frac{3B'}{4r_3^5}\right].$$

Now,

$$Q_1 q_2 + Q_2 q_1 = \rho\sin\varphi\frac{\partial W}{\partial q_1} + \rho\cos\varphi\frac{\partial W}{\partial q_2} = \rho\sin\varphi\left[\cos\varphi\frac{\partial W}{\partial \rho} - \frac{\sin\varphi}{\rho}\frac{\partial W}{\partial \varphi}\right] + \rho\cos\varphi\left[\sin\varphi\frac{\partial W}{\partial \rho} + \frac{\cos\varphi}{\rho}\frac{\partial W}{\partial \varphi}\right],$$

$$= \rho \frac{\partial W}{\partial \rho} \sin 2\varphi + \frac{\partial W}{\partial \varphi} \cos 2\varphi = 2\sqrt{-2\left(nG + C_0\right)F\left(z\right)} \sin 2\varphi + 2G\cos 2\varphi, \qquad \left[\text{using Equation (34) and (53)}\right]$$

$$= 2ane\left[-2\left(nG + C_0\right)\right]^{\frac{1}{2}} e \sin l \sin 2\varphi + 2G\cos \varphi = 2eL \sin l \sin 2\varphi + 2G\cos 2\varphi, \qquad \left[\text{using Equation (47)}\right]$$

$$Q_1 q_2 + Q_2 q_1 = 2\left[eL \sin l \sin 2\varphi + 2G \cos 2\varphi\right],$$

$$\frac{n}{2}\left(Q_1 q_2 + Q_2 q_1\right) = n\left[eL \sin l \sin 2\varphi + 2G\cos 2\varphi\right],$$

$$q_1^2 q_2^2 = \rho^2 \cos^2 \varphi \rho^2 \sin^2 \varphi = \rho^4 \left(\sin\varphi \cos\varphi\right)^2 = \frac{z^2}{4}\sin^2 2\varphi \quad \text{and} \quad q_1 q_2 = \frac{z}{2}\sin 2\varphi.$$

Thus

$$K_1 = 1 - n\left(eL \sin l \sin 2\varphi + G\cos 2\varphi\right) - z\left[C_1 + \frac{1}{r_2} + \frac{\varepsilon_0}{r_3} + \frac{\sigma_1}{2r_2^3} + \frac{A'}{r_3^3} - \frac{3B'}{4r_3^5} - \frac{B' z^2 \sin^2 2\varphi}{r_3^5} + \frac{\sqrt{3}B' z \sin 2\varphi}{2r_3^5}\right], \tag{66}$$

where $r_1 = na\left(1 - e\cos l\right) = z, \quad r_2^2 = 1 + z^2 + 2z\cos 2\varphi, \quad r_3^2 = 1 + z^2 + z\cos 2\varphi - \sqrt{3}z \sin 2\varphi$. where a is given by Equation (45), e is given by Equation (46) and φ is given by the first equation of (58).

By neglecting the higher order terms of e, let the co-efficient of μ be denoted by R then the complete Hamiltonian in terms of canonical variables l, g, L, G is given by

$$K = L\left[-2\left(nG + C_0\right)\right]^{\frac{1}{2}} - 1 + \mu R.$$

$\therefore$ The equations of motion for the complete Hamiltonian are

$$\left.\begin{aligned} \frac{dL}{d\tau} &= \frac{dK}{dl} = \mu \frac{\partial R}{\partial l}, & \frac{dG}{d\tau} &= \frac{dK}{dg} = \mu\frac{\partial R}{\partial g}, \\ \frac{dl}{d\tau} &= -\frac{dK}{dL} = -\left[-2\left(nG + C_0\right)\right]^{\frac{1}{2}} - \mu \frac{\partial R}{\partial L}, & \frac{dg}{d\tau} &= -\frac{dK}{dG} = \frac{nL}{\left[-2\left(nG + C_0\right)\right]^{\frac{1}{2}}} - \mu \frac{\partial R}{\partial G}. \end{aligned}\right\} \tag{67}$$

where

$$R = 1 - n\left(eL\sin l \sin 2\varphi + G\cos 2\varphi\right)$$
$$-z\left[G + \frac{1}{r_2} + \frac{\varepsilon_0}{r_3} + \frac{\sigma_1}{2r_2^3} - \frac{B' a^2 n^2 \left(1 - 2e\cos l\right)\sin^2 2\varphi}{r_3^5} + \frac{A'}{r_3^3} - \frac{3B'}{4r_3^5} - \frac{\sqrt{3}B' an\left(1 - e\cos l\right)\sin 2\varphi}{2r_3^5}\right].$$

The Equation (67) forms the basis of a general perturbation theory for the problem in question. The solution given in Equations (62) and (63) are periodic if l and g have commensurable frequencies that is, if

$$\left|\frac{\eta_l}{\eta_g}\right| = \frac{2\left|nG + C_0\right|}{L} = \frac{p}{q}, \tag{68}$$

where p and q are integers.

The periods of q_i, Q_i are $\frac{4\pi}{\eta_l}$ and $\frac{4\pi}{\eta_g}$, so that in case of commensurability, the period of the solution is $\frac{4\pi p}{\eta_l}$ and $\frac{4\pi q}{\eta_g}$.

V. EXISTENCE OF PERIODIC ORBITS

Here we shall follow the method used by Choudhary [4] to prove the existence of periodic orbits when $\mu \neq 0$. When $\mu = 0$, the Equations (67) become

$$\left.\begin{aligned} \frac{dL}{d\tau} &= \frac{\partial K_0}{\partial l} = 0, & \frac{dG}{d\tau} &= \frac{\partial G_0}{\partial g} = 0, \\ \frac{dl}{d\tau} &= -\frac{\partial K_0}{\partial L} = -\left[-2\left(nG + C_0\right)\right]^{\frac{1}{2}} = \eta_1\left(o\right), & \frac{dg}{d\tau} &= -\frac{\partial K_0}{\partial G} = \frac{nL}{\left[-2\left(nG + C_0\right)\right]^{\frac{1}{2}}} = \eta_2\left(o\right), \quad \text{say} \end{aligned}\right\} \tag{69}$$

Let $x_1 = L, \; x_2 = G, \; y_1 = l$ and $y_2 = g$ then

$$\frac{dx_1}{d\tau} = \frac{dx_2}{d\tau} = 0, \quad \frac{dy_1}{d\tau} = \eta_1\left(o\right), \quad \frac{dy_2}{d\tau} = \eta_2\left(o\right)$$

Thus, the Equation (69) can be written as

$$\frac{dx_{i'}}{d\tau} = 0 \quad \text{and} \quad \frac{dy_{i'}}{d\tau} = \eta_i\left(o\right)$$

$$\Rightarrow x_i = a_i, y_i = \eta_i\left(o\right)\tau + \omega_i \quad \left(i = 1, 2\right) \tag{70}$$

These are generating solutions of the two-body problem. Here a_i, η_i are constants given by

$$\eta_1(o)=\left[-\frac{\partial K_0}{\partial x_1}\right]_{x_1=a_1}, \qquad \eta_2(o)=\left[-\frac{\partial K_0}{\partial x_2}\right]_{x_2=a_2} \tag{71}$$

The generating solutions will be periodic with the period τ_0 if

$$\left.\begin{aligned} &x_i(\tau_0)-x_i(o)=0,\\ &y_i(\tau_0)-y_i(o)=\eta_i(o)\tau=2\pi\kappa_i \quad (i=1,2)\end{aligned}\right\} \tag{72}$$

Here $\kappa_i\,(i=1,2)$ are integers, so that $\eta_i(o)$ are commensurable.

Let the general solution in the neighbourhood of the generating solution be periodic with the period $\tau_0+\alpha\tau_0=(1+\alpha)\tau_0$, α is negligible quantity of the order of μ. Let us introduce new independent variable ς by the equation $\varsigma=\frac{\tau}{1+\alpha}$. The period of the general solution will be $\varsigma_0+\alpha\varsigma_0=(1+\alpha)\varsigma_0=(1+\alpha)\frac{\tau_0}{1+\alpha}=\tau_0$ which is same as the period of the generating solution. The Equation (67) now can be written as

$$\frac{dx_i}{d\varsigma}=(1+\alpha)\frac{\partial K}{\partial y_i}, \qquad \frac{dy_i}{d\varsigma}=-(1+\alpha)\frac{\partial K}{\partial x_i}. \tag{73}$$

Following Poincare [11], the general solutions in the neighbourhood of the generating solutions may be written as

$$x_i=a_i+\beta_i+\xi_i(\varsigma), \qquad y_i=\eta_i(o)\varsigma+\omega_i+\gamma_i+\eta_i(\varsigma)=\eta_i^{(o)}\varsigma+\omega_i+\gamma_i+\eta_i(\varsigma)$$

The Equation (73) can be written in terms of new variable ξ_i,η_i as

$$\frac{d\xi_i}{d\varsigma}=\frac{\partial K'}{\partial\eta_i}, \qquad \frac{d\eta_i}{d\varsigma}=-\frac{\partial K'}{\partial\xi_i} \quad (i=1,2) \tag{74}$$

where

$$\begin{aligned} K'(\varsigma,\xi_i,\eta_i)&=(1+\alpha)K\left[\varsigma,a_i+\beta_i+\xi_i,\eta_i^{(o)}\varsigma+\omega_i+\gamma_i+\eta_i\right]-(1+\alpha)K\left(\varsigma,a_i,\eta_1^{(o)}\varsigma+\omega_i\right)+\eta_1^{(o)}\xi_1+\eta_2^{(o)}\xi_2,\\ &=(1+\alpha)\left[K\left(\varsigma,a_i,\eta_i^{(o)}\varsigma+\omega_i\right)+\sum_{i=1}^{2}\left(\xi_i\frac{\partial K}{\partial a_i}+\eta_i\frac{\partial K}{\partial\omega_i}\right)\right]+\eta_1^{(o)}\xi_1+\eta_2^{(o)}\xi_2-(1+\alpha)K\left(\varsigma,a_i,\eta_1^{(o)}\varsigma+\omega_i\right),\\ &=(1+\alpha)\sum_{i=1}^{2}\left(\xi_i\frac{\partial K}{\partial a_i}+\eta_i\frac{\partial K}{\partial\omega_i}\right)+\eta_1^{(o)}\xi_1+\eta_2^{(o)}\xi_2.\end{aligned}$$

Now in order that the periodic solution may exist, the necessary and sufficient conditions are written as

$$x_i(\tau_0)-x_i(o)=\xi_i(\tau_0)=0, \tag{75}$$

$$y_i(\tau_0)-y_i(o)-2\pi\kappa_i=\eta_i(o)=0. \tag{76}$$

Restricting our solution only upto the first order infinitesimals, the equations of motion (74) may be written as

$$\frac{d\xi_i}{d\varsigma}=(1+\alpha)\frac{\partial K}{\partial\omega_i}, \tag{77}$$

$$\frac{d\eta_i}{d\varsigma}=-(1+\alpha)\frac{\partial K}{\partial\omega_i}-\eta_i^{(o)}. \tag{78}$$

Expanding $K\left(\varsigma,a_i+\beta_i,\eta_i^{(o)}\varsigma+\omega_i+\gamma_i\right)$ in ascending powers of β_i,γ_i,μ, we find that Equation (77) may be written as

$$\begin{aligned}\frac{d\xi_k}{d\varsigma}&=(1+\alpha)\frac{\partial}{\partial\omega_k}K\left(\varsigma,a_i+\beta_i,\eta_i^{(0)}\varsigma+\omega_i+\gamma_i\right),\\ &=(1+\alpha)\frac{\partial}{\partial\omega_k}\left[K_0(\varsigma,a_i+\beta_i)+\mu K_1\left(\varsigma,a_i+\beta_i,\eta_i^{(0)}\varsigma+\omega_i+\gamma_i\right)\right],\\ &=\mu\frac{\partial}{\partial\omega_k}K_1\left(\varsigma,a_i+\beta_i,\eta_i^{(0)}\varsigma+\omega_i+\gamma_i\right), \qquad [\alpha\mu \text{ is neglected}]\end{aligned}$$

$$\frac{1}{\mu}\frac{d\xi_k}{d\varsigma}=\frac{\partial}{\partial\omega_k}\left[K_1\left(\varsigma,a_i,\eta_i^{(0)}\varsigma,\omega_i\right)+\sum_{i=1}^{2}\left(\beta_i\frac{\partial K_1}{\partial a_i}+\gamma_i\frac{\partial K_1}{\partial\omega_i}\right)\right].$$

Neglecting higher order terms and integrating with respect to ς, we get

$$\frac{\xi_k(\tau_0,\beta_i,\gamma_i,\mu)}{\mu}=\int_0^{\tau_0}\left[\frac{\partial K_1}{\partial\omega_k}+\sum_{i=1}^{2}\beta_i\frac{\partial^2K_1}{\partial\omega_k\partial a_i}+\sum_{i=1}^{2}\gamma_i\frac{\partial^2K_1}{\partial\omega_k\partial\omega_i}\right]d\varsigma=\frac{\partial}{\partial\omega_k}\int_0^{\tau_0}K_1d\varsigma+\sum_{i=1}^{2}\beta_i\frac{\partial^2}{\partial\omega_k\partial a_i}\int_0^{\tau_0}K_1d\varsigma+\sum_{i=1}^{2}\gamma_i\frac{\partial^2}{\partial\omega_k\partial\omega_i}\int_0^{\tau_0}K_1d\varsigma,$$

$$\frac{\xi_k(\tau_0,\beta_i,\gamma_i,\mu)}{\mu}=\tau_0\frac{\partial[K_1]}{\partial\omega_k}+\sum_{i=1}^{2}\tau_0\beta_i\frac{\partial^2[K_1]}{\partial\omega_k\partial a_i}+\sum_{i=1}^{2}\tau_0\gamma_i\frac{\partial^2[K_1]}{\partial\omega_k\partial\omega_i},$$

where $[K_1]=\frac{1}{\tau_0}\int_0^{\tau_0}K_1\left(\varsigma,a_i,\eta_i^{(o)}\varsigma+\omega_i\right)d\varsigma.$

$$\Rightarrow \frac{\xi_k(\tau_0,\beta_i,\gamma_i,\mu)}{\mu\tau_0} = \frac{\partial[K_1]}{\partial\omega_1} + \sum_{i=1}^{2}\beta_i \frac{\partial^2[K_1]}{\partial\omega_1\partial a_i} + \sum_{i=1}^{2}\gamma_i \frac{\partial^2[K_1]}{\partial\omega_1\partial\omega_i},$$

$$\left.\begin{aligned} &\text{i.e., } \frac{\xi_k(\tau_0,\beta_i,\gamma_i,\mu)}{\mu\tau_0} = \frac{\partial[K_1]}{\partial\omega_1} + \beta_1 \frac{\partial^2[K_1]}{\partial\omega_1\partial a_1} + \beta_2 \frac{\partial^2[K_1]}{\partial\omega_1\partial a_2} + \gamma_1 \frac{\partial^2[K_1]}{\partial\omega_1^2} + \gamma_2 \frac{\partial^2[K_1]}{\partial\omega_1\partial\omega_2} = 0, \\ &\frac{\xi_k(\tau_0,\beta_i,\gamma_i,\mu)}{\mu\tau_0} = \frac{\partial[K_1]}{\partial\omega_2} + \beta_1 \frac{\partial^2[K_1]}{\partial\omega_2\partial a_1} + \beta_2 \frac{\partial^2[K_1]}{\partial\omega_2\partial a_2} + \gamma_1 \frac{\partial^2[K_1]}{\partial\omega_2\partial\omega_1} + \gamma_2 \frac{\partial^2[K_1]}{\partial\omega_2^2} = 0. \end{aligned}\right\} \tag{79}$$

From Equation (78),

$$\frac{d\eta_i}{d\varsigma} = -(1+\alpha)\frac{\partial K}{\partial a_i} - \eta_i^{(o)},$$

$$\frac{d\eta_1}{d\varsigma} = -\alpha\frac{\partial K_0}{\partial a_1} - \beta_1\sum_{i=1}^{2}\frac{\partial^2 K_0}{\partial a_1\partial a_1} - \beta_2\sum_{i=1}^{2}\frac{\partial^2 K_0}{\partial a_1\partial a_2} + o(\mu).$$

Integrating with respect to ς, we get

$$\left.\begin{aligned} \frac{\eta_1(\varsigma,\beta_i,\gamma_i,\mu)}{-\tau_0} &= \alpha\frac{\partial K_0}{\partial a_1} + \beta_1\frac{\partial^2 K_0}{\partial a_1\partial a_1} + \beta_2\frac{\partial^2 K_0}{\partial a_1\partial a_2} + o(\mu) = 0, \\ \frac{\eta_2(\varsigma,\beta_i,\gamma_i,\mu)}{-\tau_0} &= \alpha\frac{\partial K_0}{\partial a_2} + \beta_1\frac{\partial^2 K_0}{\partial a_2\partial a_1} + \beta_2\frac{\partial^2 K_0}{\partial a_2\partial a_2} + o(\mu) = 0. \end{aligned}\right\} \tag{80}$$

By implicit function theorem, we may say that ξ_1 can be expressed in terms ξ_2,η_1,η_2. So, we are left with the equations involving five variables, viz $\beta_1,\beta_2,\gamma_1,\gamma_2$ and α. Hence, two unknowns γ_1 and α may be chosen arbitrarily. Let $\gamma_1 = 0$ and $\alpha = \alpha(\mu) \neq 0$. Further the choice of the origin of time is arbitrary, so we may take $\omega_1 = 0$. The Equations (79) and (80) will give β_1,β_2,γ_2 as analytic function of μ, reducing to zero with μ, if the following conditions of Duboshin [5] are satisfied for periodic orbits.

$$\frac{\partial[K_1]}{\partial\omega_i} = 0, \qquad (i = 1,2) \tag{81}$$

$$\frac{\partial[K_1]}{\partial a_i} = 0, \qquad (i = 1,2) \tag{82}$$

$$J = \frac{\partial(\xi_2,\eta_1,\eta_2)}{\partial(\gamma_2,\beta_1,\beta_2)} \neq 0, \tag{83}$$

where $\mu = \beta_i = \gamma_i = 0$ i.e., Equations (81) and (82) together will justify Equation (83).

From Equation (83),

$$J = \begin{vmatrix} \frac{\partial\xi_2}{\partial\gamma_2} & \frac{\partial\eta_1}{\partial\gamma_2} & \frac{\partial\eta_2}{\partial\gamma_2} \\ \frac{\partial\xi_2}{\partial\beta_1} & \frac{\partial\eta_1}{\partial\beta_1} & \frac{\partial\eta_2}{\partial\beta_1} \\ \frac{\partial\xi_2}{\partial\beta_2} & \frac{\partial\eta_1}{\partial\beta_2} & \frac{\partial\eta_2}{\partial\beta_2} \end{vmatrix}$$

From Equations (79) and (80),

$$\frac{\partial\xi_2}{\partial\gamma_2} = \frac{\partial^2[K_1]}{\partial\omega_2^2}(\mu\tau_0), \qquad \frac{\partial\eta_1}{\partial\gamma_2} = 0, \qquad \frac{\partial\eta_2}{\partial\gamma_2} = 0,$$

$$\frac{\partial\xi_2}{\partial\beta_1} = \frac{\partial^2[K_1]}{\partial\omega_2\partial a_1}, \qquad \frac{\partial\eta_1}{\partial\beta_1} = -\tau_0\frac{\partial^2 K_0}{\partial a_1^2}, \qquad \frac{\partial\eta_2}{\partial\beta_1} = -\tau_0\frac{\partial^2 K_0}{\partial a_2\partial a_1},$$

$$\frac{\partial\xi_2}{\partial\beta_2} = \frac{\partial^2[K_1]}{\partial\omega_2\partial a_2}, \qquad \frac{\partial\eta_1}{\partial\beta_2} = -\tau_0\frac{\partial^2 K_0}{\partial a_1\partial a_2}, \qquad \frac{\partial\eta_2}{\partial\beta_2} = -\tau_0\frac{\partial^2 K_0}{\partial a_2^2},$$

$$J = \begin{vmatrix} \frac{\partial^2[K_1]}{\partial\omega_2^2} & 0 & 0 \\ \frac{\partial^2[K_1]}{\partial\omega_2\partial a_1} & -\tau_0\frac{\partial^2 K_0}{\partial a_1^2} & -\tau_0\frac{\partial^2 K_0}{\partial a_2\partial a_1} \\ \frac{\partial^2[K_1]}{\partial\omega_2\partial a_2} & -\tau_0\frac{\partial^2 K_0}{\partial a_1\partial a_2} & -\tau_0\frac{\partial^2 K_0}{\partial a_2^2} \end{vmatrix} = \tau_0^2\frac{\partial^2[K_1]}{\partial\omega_2^2}\begin{vmatrix} \frac{\partial^2 K_0}{\partial a_1^2} & \frac{\partial^2 K_0}{\partial a_2\partial a_1} \\ \frac{\partial^2 K_0}{\partial a_1\partial a_2} & \frac{\partial^2 K_0}{\partial a_2^2} \end{vmatrix},$$

$$J=\tau_0^2\frac{\partial^2[K_1]}{\partial\omega_2^2}\left[\frac{\partial^2K_0}{\partial a_1^2}.\frac{\partial^2K_0}{\partial a_2^2}-\left(\frac{\partial^2K_0}{\partial a_1\partial a_2}\right)^2\right].$$

Now,

$$K_0=a_1\left[-2\left(na_2+C_0\right)\right]^{\frac{1}{2}}-1,$$

$$\frac{\partial K_0}{\partial a_1}=a_1\left[-2\left(na_2+C_0\right)\right]^{\frac{1}{2}},\qquad \frac{\partial^2K_0}{\partial a_1^2}=0,$$

$$\frac{\partial^2K_0}{\partial a_1\partial a_2}=\frac{-n}{\left[-2\left(na_2+C_0\right)\right]^{\frac{1}{2}}}\qquad\Rightarrow\left(\frac{\partial^2K_0}{\partial a_1\partial a_2}\right)^2=\frac{n^2}{-2\left(na_2+C_0\right)},$$

$$J=\tau_0^2\frac{\partial^2[K_1]}{\partial\omega_2^2}\times\frac{n^2}{2\left(na_2+C_0\right)}=\frac{n^2\tau_0^2}{2\left(na_2+C_0\right)}.\frac{\partial^2[K_1]}{\partial\omega_2^2},$$

$$J=\frac{n^2\tau_0^2}{2\left(na_2+C_0\right)}.\frac{\partial^2[K_1]}{\partial\omega_2^2}.\tag{84}$$

Now let us find $\frac{\partial[K_1]}{\partial a_i},\frac{\partial[K_1]}{\partial\omega_i}\quad(i=1,2)$. Taking only zero-degree terms (i.e. for $e=0,\ f=l=y_1$)

$$r_1=na=z,\qquad r_2^2=1+n^2a^2+2na\cos2\varphi,$$

$$r_3^2=1+n^2a^2+na\cos2\varphi-\sqrt{3}na\sin2\varphi=1+n^2a^2+2na\cos\left(2\varphi+\frac{\pi}{3}\right),$$

$$2\varphi=y_1+y_2-\frac{n\sqrt{x_1^2-x_2^2}}{2\left(nx_2+C_0\right)}\sin y_1,$$

$$y_1=\eta_1^{(o)}\varsigma+\omega_1+\gamma_1+\eta_1(\varsigma),\qquad y_2=\eta_2^{(o)}\varsigma+\omega_2+\gamma_2+\eta_2(\varsigma),$$

$$x_1=a_1+\beta_1+\xi_1(\varsigma),\qquad x_2=a_2+\beta_2+\xi_2(\varsigma),$$

$$\frac{\partial r_2}{\partial\omega_i}=-\frac{na\sin2\varphi}{r_2}\left(\frac{2\partial\varphi}{\partial\omega_i}\right),\qquad \frac{\partial r_3}{\partial\omega_i}=-\frac{na\sin\left(2\varphi+\frac{\pi}{3}\right)}{r_3}\left(\frac{2\partial\varphi}{\partial\omega_i}\right),\qquad(i=1,2)$$

$$[K_1]=1-nG\cos2\varphi-na\left[C_1+\frac{1}{r_2}+\frac{\varepsilon_0}{r_3}+\frac{\sigma_1}{2r_2^3}+\frac{A'}{r_3^3}-\frac{3B'}{4r_3^5}-\frac{B'a^2n^2\sin^22\varphi}{r_3^5}+\frac{\sqrt{3}B'an\sin2\varphi}{2r_3^5}\right].\tag{85}$$

$$\frac{\partial[K_1]}{\partial\omega_i}=nG\sin2\varphi\left(2\frac{\partial\varphi}{\partial\omega_i}\right)-na\left[-\frac{1}{r_2^2}\frac{\partial r_2}{\partial\omega_i}-\frac{3\sigma_1}{2r_2^4}\frac{\partial r_2}{\partial\omega_i}-\frac{\varepsilon_0}{r_3^2}\frac{\partial r_3}{\partial\omega_i}-\frac{3A'}{r_3^4}\frac{\partial r_3}{\partial\omega_i}+\frac{15B'}{4r_3^6}\frac{\partial r_3}{\partial\omega_i}\right.$$
$$\left.-B'n^2a^2\left(\frac{r_3^5\,2\sin2\varphi\cos2\varphi2\frac{\partial\varphi}{\partial\omega_i}-\sin^22\varphi5r_3^4\frac{\partial r_3}{\partial\omega_i}}{r_3^{10}}\right)+\frac{\sqrt{3}B'na}{2}\left(\frac{r_3^5\cos2\varphi2\frac{\partial\varphi}{\partial\omega_i}-\sin2\varphi5r_3^4\frac{\partial r_3}{\partial\omega_i}}{r_3^{10}}\right)\right],$$

$$\left.\begin{aligned}\frac{\partial[K_1]}{\partial\omega_i}=&\left(2\frac{\partial\varphi}{\partial\omega_i}\right)\left[nG\sin2\varphi-\frac{n^2a^2}{r_2^3}\sin2\varphi-\frac{3n^2a^2\sigma_1}{2r_2^5}\sin2\varphi-\frac{3A'n^2a^2}{2r_3^5}-\frac{\varepsilon_0n^2a^2}{r_3^3}\sin\left(2\varphi+\frac{\pi}{3}\right)\right.\\&+\frac{B'n^3a^3}{r_3^5}2\sin2\varphi\cos2\varphi-\frac{\sqrt{3}B'n^2a^2}{2r_3^5}\cos2\varphi+\frac{5B'n^4a^4}{r_3^7}\sin^22\varphi\sin\left(2\varphi+\frac{\pi}{3}\right)+\frac{15B'n^2a^2}{4r_3^7}\sin\left(2\varphi+\frac{\pi}{3}\right)\\&\left.-\frac{5\sqrt{3}B'n^3a^3}{2r_3^7}\sin2\varphi\sin\left(2\varphi+\frac{\pi}{3}\right)\right],\end{aligned}\right\}\tag{86}$$

$$\Rightarrow\frac{\partial[K_1]}{\partial\omega_i}=\left(2\frac{\partial\varphi}{\partial\omega_i}\right)N,\qquad\text{Similarly}\ \frac{\partial[K_1]}{\partial a_i}=\left(2\frac{\partial\varphi}{\partial a_i}\right)\mathrm{N}\quad(i=1,2)\tag{87}$$

where

$$\begin{aligned}N=&\left[nG\sin2\varphi-\frac{n^2a^2}{r_2^3}\sin2\varphi-\frac{3n^2a^2\sigma_1}{2r_2^5}\sin2\varphi-\frac{3A'n^2a^2}{2r_3^5}-\frac{\varepsilon_0n^2a^2}{r_3^3}\sin\left(2\varphi+\frac{\pi}{3}\right)+\frac{B'n^3a^3}{r_3^5}2\sin2\varphi\cos2\varphi\right.\\&\left.-\frac{\sqrt{3}B'n^2a^2}{2r_3^5}\cos2\varphi+\frac{5B'n^4a^4}{r_3^7}\sin^22\varphi\sin\left(2\varphi+\frac{\pi}{3}\right)+\frac{15B'n^2a^2}{4r_3^7}\sin\left(2\varphi+\frac{\pi}{3}\right)-\frac{5\sqrt{3}B'n^3a^3}{2r_3^7}\sin2\varphi\sin\left(2\varphi+\frac{\pi}{3}\right)\right].\end{aligned}\tag{88}$$

Here $\frac{\partial[K_1]}{\partial\omega_i}=\frac{\partial[K_1]}{\partial a_i}=0$ if and only if $N=0$, because $\frac{\partial\varphi}{\partial\omega_i},\frac{\partial\varphi}{\partial a_i}(i=1,2)$ are not necessarily zero simultaneously. For making $N=0$, putting $\cos 2\varphi=0$ i.e., $2\varphi=\frac{\pi}{2}$ i.e., $\sin 2\varphi=1$ and $\sin\left(2\varphi+\frac{\pi}{3}\right)=\frac{1}{2}$.

$$nG-\frac{n^2a^2}{r_2^3}-\frac{3n^2a^2\sigma_1}{2r_2^5}-\frac{\varepsilon_0 n^2a^2}{2r_3^3}-\frac{3A'n^2a^2}{2r_3^7}+\frac{5B'n^4a^4}{2r_3^7}+\frac{15B'n^2a^2}{8r_3^7}-\frac{5\sqrt{3}B'n^3a^3}{4r_3^7}=0,$$

$$\left.\begin{aligned}\Rightarrow G&=na^2\left[\frac{1}{r_2^3}+\frac{3\sigma_1}{2r_2^5}+\frac{\varepsilon_0}{2r_3^3}+\frac{3A'}{2r_3^5}+\frac{5B'n^2a^2}{2r_3^7}-\frac{5\sqrt{3}B'na}{4r_3^7}+\frac{15B'}{8r_3^7}\right],\\ &=na^2\left[\frac{1}{r_2^3}+\frac{3\sigma_1}{2r_2^5}+\frac{\varepsilon_0}{2r_3^3}+\frac{3A'}{2r_3^5}+\frac{5B'}{8r_3^7}\left(4n^2a^2-2\sqrt{3}na+3\right)\right],\end{aligned}\right\} \tag{89}$$

where the parameters $\sigma_1,n,a,\varepsilon_0,A',B'$ are given in Equation (31) of previous section.

Now from Equation (87),

$$\frac{\partial[K_1]}{\partial\omega_2}=2\frac{\partial\varphi}{\partial\omega_2}N,$$

$$\frac{\partial^2[K_1]}{\partial\omega_2^2}=2\left[\frac{\partial^2\varphi}{\partial\omega_2^2}N+\frac{\partial\varphi}{\partial\omega_2}\frac{\partial N}{\partial\omega_2}\right]=2\frac{\partial^2\varphi}{\partial\omega_2^2}N+2\frac{\partial\varphi}{\partial\omega_2}\frac{\partial N}{\partial\omega_2}=\frac{\partial N}{\partial\omega_2},\qquad\left[\text{as }\frac{\partial\varphi}{\partial\omega_2}=\frac{1}{2}\right]$$

$$\frac{\partial^2[K_1]}{\partial\omega_2^2}=-\left[\frac{3n^3a^3}{2r_2^7}\left(5\sigma_1+r_2^2\right)+\frac{n^3a^3}{16r_3^9}\left(12\varepsilon_0r_3^4+60A'r_3^2-140B'n^2a^2-105B'+70\sqrt{3}B'na\right)\right].$$

By putting suitable values of all the parameters in the right hand side of Equation (90), $\frac{\partial^2[k_1]}{\partial\omega_2^2}\neq 0$ i.e., $J\neq 0$ i.e., the conditions of the existence of periodic orbits given by Duboshin [5] are satisfied. Thus, the periodic orbits of the infinitesimal mass about any primary are periodic.

VI. CONCLUSIONS

In order to prove the existence of periodic orbits of the first kind in the Circular Restricted Four-body Problem, we have discussed the problem into five sections starting with introduction about the historical evolution of the topic. In the second section, we established the equations of motion of the infinitesimal mass under the perturbed gravitational field of the three primaries. In the present problem, the second primary is an oblate spheroid and third primary is a tri-axial rigid body. All the primaries are moving on circular orbits about the centre of mass of the dominant primaries P_1 and P_2. The primaries P_1 and P_2 are dominant in the sense that P_1 and P_2 have influence of attraction on the third primary P_3 and infinitesimal mass P but P_3 and P have no influence of attraction on the primaries P_1 and P_2 whereas P_3 has an influence of attraction on the infinitesimal mass P only but not on P_1 and P_2. That's the reason; the centre of mass P_1 and P_2 didn't change. The second section ended with the energy integral of the infinitesimal mass at $P(x_1,x_2)$.

The energy function H contains three singularities $r_1=0, r_2=0$ and $r_3=0$ so in Hamiltonian, mechanics to keep the energy function $H=\text{constant}$, we need to eliminate any singularity for the case of collision with the corresponding primary. In the third section, we have introduced a suitable generating function for regularization of H to eliminate the singularity at $r_1=0$. After regularizing the Hamiltonian $H=C$, we have developed the canonical equations of motion corresponding to the regularized Hamiltonian $K=0$.

In fourth section, we have established the generating solution i.e., the solutions of the equations of motion of the infinitesimal mass by taking the first primary at the origin i.e., at the centre of mass. On this consideration, we got $\mu=0$ and the Hamiltonian becomes K_0. By taking K_0 as our Hamiltonian, we get the solution of the equations of motion, which is called generating solution. With the help of generating solution and the method of analytic continuation, we can find the general solution corresponding to the complete Hamiltonian $K=K_0+\mu K_1$ where $\mu\neq 0$.

In fifth section, we have examined the existence of periodic orbits when $\mu\neq 0$ with the technique of Chaudhary [4] applying to the conditions given by Duboshin [5]. Since our consideration satisfied all the conditions for periodic orbits given by Duboshin, hence we conclude that the periodic orbits of the infinitesimal mass around the first primary exist when suitable values of μ,σ_1,σ_2 are taken. By shifting the origin to the centre of the other primaries also, the existence of periodic orbits can be examined. Even by using "Mathematica", we can show the existence of periodic orbits of the infinitesimal mass around other primaries also, by taking suitable values of the parameters.

REFERENCES

[1] M. R. Hassan, Md. Aminul Hassan, Payal Singh, Vinay Kumar, R. R. Thapa, "Existence of Periodic Orbits of the First kind in the Autonomous Four – body Problem with the Case of Collision", *International Journal of Astronomy and Astrophysics*, vol. 7, pp. 91 – 111, June 2017.

[2] M. R. Hassan, Payal Singh, Md. Aminul Hassan, "Periodic Orbits of the First kind in the CR4BP when the Second Primary is a Triaxial Rigid body", *International Journal of Astronomy and Astrophysics*, vol. 7, pp. 133 – 153, July 2017.

[3] Payal Singh, M. R. Hassan, Md. Aminul Hassan, "Existence of Periodic Orbits of the First kind in the CR4BP when the Second and Third Primaries are Triaxial Rigid bodies", *International Journal for Research in Engineering Application and Management* (*IJREAM*), vol. 06, issue 07, pp. 01 – 13, October 2020.

[4] R. K. Choudhary, "Existence of Periodic Orbits of the Third kind in the Elliptic Restricted Problem of the Three bodies and the Stability of the Generating Solution", *Proceedings of the National Academy of Sciences*, vol. 36a, pp. 249 – 264, 1966.

[5] G. N. Duboshin, "Analytical and Qualitative Methods" (Russian), *Celestial Mechanics*, pp. 178 – 184, 1964.

[6] E. O. Giacaglia, "Periodic Orbits of Collision in the Restricted Problem of Three bodies", *Astronomical Journal*, vol. 72, pp. 386 – 391, 1967.

[7] K. B. Bhatnagar, "Periodic Orbits of Collision in the Plane Elliptic Restricted Problem of Three bodies", *National Institute of Science*, vol 35a, pp. 829 – 844, 1969.

[8] K. B. Bhatnagar, "Periodic Orbits of Collision in the Plane Circular Problem of Four bodies", *Indian Journal of Pure and Applied Mathematics*, vol 2, pp. 583 – 596, 1971.

[9] M. Ceccaroni, J. Biggs, "Low – Thrust Propulsion in a Coplanar Circular Restricted Four – body Problem", *Celestial Mechanics and Dynamical Astronomy*, vol. 112, pp. 191 – 219, 2012.

[10] S. W. McCuskey, "Introduction to Celestial Mechanics", *Addison – Wesley Publication Corp*, 1963.

[11] H. Poincare, "Lecons de Mecanique Celeste", Gauthier – Villars, 1, 1905.