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Describe the motion of N-bodies Problem

Dr. Chandan Kumar Singh

PG Dept. of Mathematics TMBU Bhagalpur,chandandadhiwala@gmail.com

***Abstract*— This paper deals with N-bodies Problems which are moving under the Newtonian law and find its equations. Solve the Two body problem and General three body problem and we get six integrals for centre of mass, three area integrals and one integral for energy. We describe the Classification of Motion and special solution.**

***Keywords*— Two body problem, The Solar system, N-bodies problem, General Three body problem, Classification of motion.**

I. INTRODUCTION

Dynamics is an interdisciplinary subject today, it was originally a branch of physics. The subject of dynamics was found by Galileo in the earlier period of seventeenth century and developed further by Huygens after Galileo. The fundamental principles were clarified and crystallized by Newton as a part of Celestial Mechanics.Celestial mechanics is concerned with the motions of the objects in astronomical space, with the physical forces that govern these motions, and with the mathematical assumptions, condition and process by which they are determined and predicted for observation and correction. According to Gaul, the celestial mechanics is divided into three parts (Herrick, 1971). Aryabhata (476-550 CE) was the first in the line of great mathematician-astronomer from the classical age of Indian mathematics and Indian astronomy Aryabhata's system of astronomy was called the audAyaka System. Aryabhata appears to have believed that the earth rotates about its axes. He described a geocentric model of the solar system, in which the Sun and the Moon are each carried by epicycles. They in turn revolved around the Earth. In this model, which is also found in the Paitamahasiddhanta (C.E. 425).The motions of planets are each governed by the two epicycles, a smaller manda (slow) and a large Sighara (fast). The order of the planets in terms of the distances from the earth is taken as: the Moon, Mercury, Venus, the Sun. Mars, Jupiter, Saturn and the asterisms. Aryabhata states that the shines by the reflected sunlight. In seventeenth century, Newton (1687) proved that a simple inverse square law of force gives rise to all motion in the solar system and Kepler's laws of motion are natural consequences of this force and that the resulting motions are described by Conic sections. Therefore the key discovery in this process was Newton's formulation of universal law of gravitation. Using this universal law, we shall discuss some configurations of the universe.

II. TWO BODY PROBLEM

The wide variety of masses in solar system permits the orbit of most planets and satellite to be approximated by two body motions, consisting of smaller body moving around a much larger central body. The path of Jupiter (mass $m_j = 1.9\times10^{27}$ kg) around the sun (mass $m_s = 2.0\times10^{30}$ kg) is basically an ellipse with principal perturbation coming from other planets.

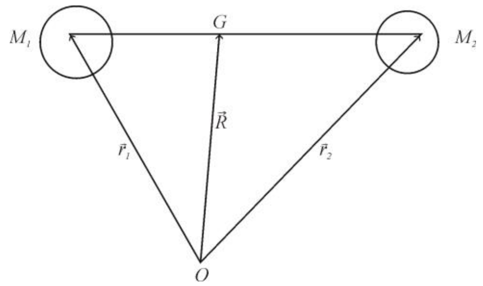


Figure : Two body Problem

Let M_1 and M_2 be two spherical bodies with masses m_1 and m_2 respectively. Let O be the origin, $\overrightarrow{OM}_1 = \vec{r}_1$ and $\overrightarrow{OM}_2 = \vec{r}_2$ .Two masses attract each other according to the Newton's law of gravitation.Force experienced by M_1 and M_2 be

$$m_1\ddot{\vec{r}}_1 = K\frac{m_1m_2}{\left|\overrightarrow{M_1M_2}\right|^3}\overrightarrow{M_1M_2} \quad ...(1)$$

$$m_2\ddot{\vec{r}}_2 = K\frac{m_2m_1}{\left|\overrightarrow{M_2M_1}\right|^3}\overrightarrow{M_2M_1} \quad ...(2)$$

Adding Equations (1) and (2), we get

$$m_1\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}_2 = \vec{0}$$

Integrating $m_1\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2 = \vec{c}$ (A constant vector) ...(3)

Again, integrating $m_1\vec{r}_1 + m_2\vec{r}_2 = \vec{c}t + \vec{d}$ ...(4)

Let $m_1 + m_2 = M$, $m_1\vec{r}_1 + m_2\vec{r}_2 = M\vec{R}$. ...(5)

$\vec{R}$ is called the position vector of centre of mass. The Equation (4) is written as

$$M\vec{R} = \vec{c}t + \vec{d}$$

$$\Rightarrow \vec{R} = \frac{1}{M}(\vec{c})t + \frac{1}{M}\vec{d} \quad ...(6)$$

Equation (6) indicates that centre of mass moves in a straight line. Differentiating both sides of Equation (5) with respect to t, we get

$$m_1\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2 = M\dot{\vec{R}} \quad ...(7)$$

From Equations (3) and (7), we get

$$M\dot{\vec{R}} = \vec{c}$$

$$\Rightarrow \dot{\vec{R}} = \frac{1}{M}\vec{c} \quad ...(8)$$

This shows that the centre of mass moves with uniform velocity. Let us choose centre of mass as origin then from Equations (1) and (2), we get

$$m_1\frac{d^2\overrightarrow{GM_1}}{dt^2} = K\frac{m_1m_2}{\left|\overrightarrow{M_1M_2}\right|^3}\overrightarrow{M_1M_2} \quad ...(9)$$

$$m_2\frac{d^2\overrightarrow{GM_2}}{dt^2} = K\frac{m_2m_1}{\left|\overrightarrow{M_2M_1}\right|^3}\overrightarrow{M_2M_1} \quad ...(10)$$

In this case, $\vec{R} = \vec{0}$.

∴ From Equation (5), we get,

$$m_1\vec{r}_1 + m_2\vec{r}_2 = \vec{0}$$

$$\Rightarrow m_1\overrightarrow{GM}_1 + m_2\overrightarrow{GM}_2 = \vec{0}$$

$$\Rightarrow m_1\overrightarrow{GM}_1 = -m_2\overrightarrow{GM}_2$$

$$\Rightarrow m_1\overrightarrow{GM}_1 + m_2\overrightarrow{GM}_1 = -m_2\overrightarrow{GM}_2 + m_2\overrightarrow{GM}_1$$

$$\Rightarrow (m_1 + m_2)\overrightarrow{GM}_1 = -m_2\overrightarrow{GM}_2 - m_2\overrightarrow{M_1G}$$

$$\Rightarrow M.\overrightarrow{GM}_1 = -m_2\overrightarrow{M_1M_2}$$

$$\Rightarrow \overrightarrow{GM}_1 = -\frac{m_2}{M}\overrightarrow{M_1M_2} \qquad \text{...(11)}$$

Similarly,

$$\Rightarrow \overrightarrow{GM}_2 = -\frac{m_1}{M}\overrightarrow{M_2M_1} \qquad \text{...(12)}$$

Practically, centre of mass cannot be taken as origin because one has to observe the motion of a body from another body. Let an observer observe motion of M_2 from M_1. Therefore, M_1 will be the origin i.e., the motion of M_2 will be calculated relative to M_1.Using (11) in (9), we get

$$\frac{d^2}{dt^2}\left(\overrightarrow{M_1M_2}\right) = -K\frac{M}{\left|\overrightarrow{M_1M_2}\right|^3}\overrightarrow{M_1M_2} \qquad \text{...(13)}$$

Then Equation (13) is written as

$$\frac{d^2\vec{r}}{dt^2} = -K\frac{M}{|\vec{r}|^3}\vec{r} \qquad \text{...(14)}$$

This equation is an equation of central force field. So, we may describe easily the entire characteristics of the motion which fulfills the Kepler's three laws.

III. The Solar system

The observable universe gives the appearance of being composed of a large number of galaxies, or stars cluster, each containing billions of stars. Many, including our own, rotate like giant pinwheels, are called the spiral galaxies. Our sun is one of the stars in the Milky Way. Its nearest neighbour is Alpha Centauri, which lies about four light years away. The sun possesses a system of satellites consisting of nine major planets, numerous minor planets and asteroids, comets, and streams of particles and cosmic "dust". This assemblage is known as the solar system.

IV. N-BODIES PROBLEM

The basic ideas of N – Body problem were published in 1687 by Sir Isaac Newton in his Principia. The limitations to his work were given later by Henry Poincare, who described the non-integrability principle as applicable to problems of three and more bodies.The only known analytical solutions of the N-body problem are the Euler and Lagrange solution that exist for any number of bodies and any mass ratios.

Statement: To describe the motion of N-bodies which are moving under the Newtonian law of gravitation is called the N-body problem. In mathematical language it means given the position vector $\bar{x}_0$ and velocity vector $\dot{\bar{x}}_0$ at time t_0, to determine the position vector $\bar{x}$ and the velocity vector $\dot{\bar{x}}$ at time t i.e.,

$$\bar{x} = x_i\left(t, t_0, \bar{x}_0, \dot{\bar{x}}_0\right), \ldots \qquad (15)$$
$$\dot{\bar{x}} = \dot{x}_i\left(t, t_0, \bar{x}_0, \dot{\bar{x}}_0\right).$$

It is quite natural to ask whether for any value of N or for any shape of bodies tae problem has been solved or not. Our answer is both yes or no. The answer is no because even for N=2, we cannot describe the motion in the mathematical form (15). And the answer is Yes, because for N-2, we agree that the problem has been solved when the bodies are point masses or spherical in shape though strictly not in the mathematical form (15). The well-known solution is

$M = E - eSinE,$ $\qquad$ $M = n(t - \tau),$

$$\theta = \omega + Cos^{-1}\left(\frac{CosE - e}{1 - eCosE}\right),$$

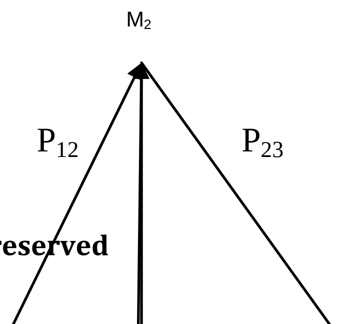
$x = aCosE,$ $\qquad$ $y = bSinE,$

$$\alpha_1 = -\frac{\mu}{2a}, \qquad \beta_1 = -\tau,$$

V.GENERAL THREE BODY PROBLEM

The problem of three bodies is defined as follows: three particles with arbitrary masses attract each other according to Newtonian law of gravitation; they are free to move in space and are initially moving in any given manner. To find their subsequent motion is known as the general three body problem. In general the three masses are point masses or spherical in shape. We can take other shape as well. If beside the gravitational forces, such as solar radiation pressure or aero dynamical forces, acting, these can also be taken as perturbations.Some of the major contributors to the three body problems are Euler and Lagrange [1772], Jacobi [1836], Hill [1878], Poincare [1892-99], Whittaker [1904], Levi-Civita [1904], and Birkhoff [1915].

Let M_1, M_2 and M_3 be three bodies shown in the figure.



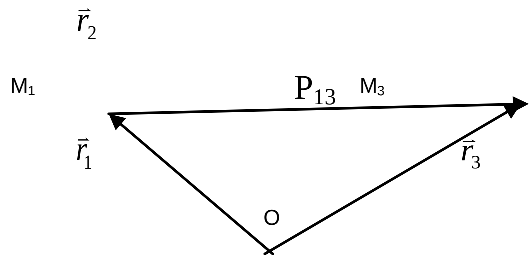


Figure : Three Body Problem

Similar to two body problem, the force experienced by three bodies be

$$m_1\ddot{\vec{r}}_1 = K\left[\frac{m_1 m_2}{P_{12}^3}\vec{P}_{12} + \frac{m_1 m_3}{P_{13}^3}\vec{P}_{13}\right] \quad ...(16)$$

$$m_2\ddot{\vec{r}}_2 = K\left[\frac{m_2 m_1}{P_{21}^3}\vec{P}_{21} + \frac{m_2 m_3}{P_{23}^3}\vec{P}_{23}\right] \quad ...(17)$$

$$m_3\ddot{\vec{r}}_3 = K\left[\frac{m_3 m_1}{P_{31}^3}\vec{P}_{31} + \frac{m_3 m_2}{P_{32}^3}\vec{P}_{32}\right] \quad ...(18)$$

Here, we need to find eighteen integrals.Adding Equations (16), (17) and (18), we get

$$m_1\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}_2 + m_3\ddot{\vec{r}}_3 = \vec{0} \quad ...(19)$$

Integrating, $m_1\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2 + m_3\dot{\vec{r}}_3 = \vec{c}$ ...(20)

Again integrating, $m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 = \vec{c}t + \vec{d}$ ...(21)

Let $m_1 + m_2 + m_3 = M$ and $M\vec{R} = m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3$ ...(22)

Then from Equations (21) and (22), we get

$$M\vec{R} = \vec{c}t + \vec{d}$$

$$\Rightarrow \vec{R} = \frac{1}{M}(\vec{c})t + \frac{1}{M}\vec{d} \quad ...(23)$$

This shows that centre of mass moves in a straight line. Differentiating both sides of Equation (22) with respect to t, we get

$$M\dot{\vec{R}} = m_1\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2 + m_3\dot{\vec{r}}_3 \quad ...(24)$$

From Equations (20) and (22), we get

$$M\dot{\vec{R}} = \vec{c}$$

$$\Rightarrow \dot{\vec{R}} = \frac{1}{M}(\vec{c})$$

Hence, the centre of mass moves in uniform motion. Taking vector products in equations (16), (17) and (18) and then adding, we get

$$m_1(\vec{r}_1 \times \ddot{\vec{r}}_1) + m_2(\vec{r}_2 \times \ddot{\vec{r}}_2) + m_3(\vec{r}_3 \times \ddot{\vec{r}}_3) = \vec{0}$$

$$m_1 \frac{d}{dt}(\vec{r}_1 \times \dot{\vec{r}}_1) + m_2 \frac{d}{dt}(\vec{r}_2 \times \dot{\vec{r}}_2) + m_3 \frac{d}{dt}(\vec{r}_3 \times \dot{\vec{r}}_3) = \vec{0}$$

$$\Rightarrow m_1(\vec{r}_1 \times \dot{\vec{r}}_1) + m_2(\vec{r}_2 \times \dot{\vec{r}}_2) + m_3(\vec{r}_3 \times \dot{\vec{r}}_3) = \vec{a} \qquad ...(25)$$

This shows that moment of momentum or angular momentum of three- body problem is always constant. Hence three bodies move in such way that the total angular momentum vector is always towards a fixed direction and the plane perpendicular to this vector is always invariable. This invariable plane is called the invariable plane of Laplace. Equation (24) is written as

$$\therefore m_1\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & y_1 & z_1 \\ \dot{x}_1 & \dot{y}_1 & \dot{z}_1 \end{vmatrix} + m_2\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_2 & y_2 & z_2 \\ \dot{x}_2 & \dot{y}_2 & \dot{z}_2 \end{vmatrix} + m_3\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_3 & y_3 & z_3 \\ \dot{x}_3 & \dot{y}_3 & \dot{z}_3 \end{vmatrix} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \qquad ...(26)$$

a_1, a_2 and a_3 are called area integrals.

Taking scalar product in Equations (16), (17) and (18), we get

$$m_1(\dot{\vec{r}}_1 \bullet \ddot{\vec{r}}_1) + m_2(\dot{\vec{r}}_2 \bullet \ddot{\vec{r}}_2) + m_3(\dot{\vec{r}}_3 \bullet \ddot{\vec{r}}_3) = -K\left[\frac{m_1m_2}{P_{12}^3}\dot{\vec{P}}_{12}\vec{P}_{12} + \frac{m_1m_3}{P_{13}^3}\dot{\vec{P}}_{13}\vec{P}_{13} + \frac{m_2m_3}{P_{23}^3}\dot{\vec{P}}_{23}\vec{P}_{23}\right]$$

$$= -K\left[\frac{m_1m_2}{P_{12}^3}\dot{P}_{12}P_{12} + \frac{m_1m_3}{P_{13}^3}\dot{P}_{13}P_{13} + \frac{m_2m_3}{P_{23}^3}\dot{P}_{23}P_{23}\right]$$

$$\Rightarrow m_1\frac{d}{dt}\left(\frac{1}{2}\dot{\vec{r}}_1^{\,2}\right) + m_2\frac{d}{dt}\left(\frac{1}{2}\dot{\vec{r}}_2^{\,2}\right) + m_3\frac{d}{dt}\left(\frac{1}{2}\dot{\vec{r}}_3^{\,2}\right)$$

$$= K\left[m_1m_2\frac{d}{dt}\left(\frac{1}{P_{12}}\right) + m_1m_3\frac{d}{dt}\left(\frac{1}{P_{13}}\right) + m_2m_3\frac{d}{dt}\left(\frac{1}{P_{23}}\right)\right]$$

$$\Rightarrow \frac{1}{2}\left(m_1\dot{\vec{r}}_1^{\,2} + m_2\dot{\vec{r}}_2^{\,2} + m_3\dot{\vec{r}}_3^{\,2}\right) - K\left[\frac{m_1m_2}{P_{12}} + \frac{m_1m_3}{P_{13}} + \frac{m_2m_3}{P_{23}}\right] = h \qquad ...(27)$$

h is called energy integral.

Hence, we get six integrals for centre of mass, three area integrals and one integral for energy. Therefore, total number of integrals obtained = 10. So, remaining eight integrals are to be derived. Force function for three body problem is a function of distance between particles only. So, this enables us to reduce the order by unity. Time t does not contain explicitly, which enables us to reduce the order by one.So, remaining 8-2 = 6 integrals are to be derived.

VI. CLASSIFICATION OF MOTION

Tapley and Szebehely (1973) have classified various type of motion and the same has been reviewed by Bhatnagar (1990). We may state briefly as follows:

When $h > 0,\ l = \infty,\ h = \frac{1}{2}\sum_{i=1}^{3} m_i v_i^2 - U$ and $I = \sum_{i=1}^{3} m_i r_i^2$, the motion could be hyperbolic – explosion, hyperbolic or parabolic orbits and hyperbolic-elliptic (binary). When *h* <0and *I* is bounded the motion could be either interplay or revolution or periodic.When *h* <0and $I \to \infty$, then the motion could be hyperbolic – elliptic (binary) or parabolic-elliptic (binary).We know that the escape orbits are dense; interplay leads to escape or ejection and repeated ejection leads to escape.

Special Solution: Following are some of the important results regarding three – body problem:

(i). Lagrange's solutions: There are five well known Lagrange's solutions–three collinear L_i (i = 1, 2, 3) and two triangularL_i (*i* = 4, 5).

(ii). Sundman result: It states that three – body motions with a non-zero angular momentum cannot approach a triple collision.

(iii). There are first 10 integrals; 6 from motion of the centre of mass; 3 from angular momentum and 1 from energy.

(iv). Lower bond of the semi- moment of inertia

$$I = \frac{1}{2}\sum_{i=1}^{3} m_i r_i^2 \quad ...(1.28)$$

can be given in term of the integrals of the motion.

(v). Chazy conjectures: Chazy has conjectured several hypotheses for the three body problem.

(a). For $h > 0,\ \ \bar{c} \neq 0,\ \ m_1, m_2, m_3 (\neq 0)$ there exist 49 possible combinations of original and final evolution. This is true even for large values of *h* and $\bar{c}$.

(b). For types, $h > 0,\ \ \bar{c} \neq 0,$ there are nine possible modes of evaluation including two oscillatory types.

Chazy also conjectured that *h* <0 the motions of exchange type are impossible. The alternative to this conjecture is the existence of a complex cantor set structure for the exchange motions. The second conjecture was challenged by (Schmidt 1947; Khime 1961; Alekseev 1965) and a numerical example presented by Alekseev remained inconclusive. However Szebehely (1975) computed a simple and symmetrical exchange motion for the masses $m_1=1,\ m_2 = m_3 = 2$. The Second Chazy conjecture remains partially true. According to Marchal (1988) the final evolution of three body problem have classified by Chazy in (1922) with some minor improvements (Table 1).

Table 1 : Evolution of Three body Problem

Class	**Type**	**Conditions on $\bar{c}$ (angular momentum) or *h* (energy)**	**Evolutions of R(=sup r_{ij}) and r(= infr_{ij})**
Singular type Hyperbolic expansions	Triple collision at t_c	$\bar{c}$ = 0	R and $r \sim (t_c - t)^{\frac{2}{3}}$
	Hyperbolic type	*h* >0	$r \sim t$

	Hyper-parabolic type	$h > 0$	$R \sim t, r \sim t^{\frac{2}{3}}$
	Hyper-elliptic Type	$h >< 0$	r bounded
Parabolic expansions Sub-parabolic	Tri-Parabolic type	$h = 0$	$R \sim t^{\frac{2}{3}}, r \sim t^{\frac{2}{3}}$
	Para-elliptic type	$h < 0$	r bounded
Sub-parabolic	Bounded type	$h < 0$	$0 < b \leq r \leq R \leq B < \infty$
Types	Oscillatory type-I	$h < 0$	*Lim Inf* $R < \infty$, r abounded
	Oscillatory type-II	$h < 0$	R bounded, *Lt r* inf= 0

V. Conclusions

In the N-bodies problem which are moving under the Newtonian law of gravitation i.e. given the position vector $\bar{x}_0$ and velocity vector $\dot{\bar{X}}_0$ at time t_0, to determine the position vector $\bar{X}$ and the velocity vector $\dot{\bar{X}}$ at time *t, which may be solvable or not as value of N- or N=2. When describe three body problem this show that, moment of momentum of three body problem is always constant and we get six integrals for centre of mass, three area integrals and one integrals of energy. Therefore, total number of integrals obtained = 10, So remaining eight integrals are to be derived. Force function for three body problem is a function of distance between particles only. So, this enables us to reduce the order by one. So, remaning 8 - 2 = 6 integrals are to be derived. Scientist tried for a long time to find these remaining integrals but in vain. So three body problem is unsolvable.*

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