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On Certain Results Involving Bilateral Basic Hypergeometric Functions And Infinite Products

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Abstract : Some unilateral basic hypergeometric functions can be expressed in the form of bilateral basic hypergeometric functions and infinite products. In this paper, an attempt has been made to establish the results involving bilateral basic hypergeometric function and infinite products using well known Heine's transformation and summation formula and q-analog of Euler's transformation and summation formula.

Keywords :Bilateral Basic Hypergeometric Functions , Infinite products.

1. Introduction

In 2014, Bindu Prakash Mishra[2] established certain transformations for unilateral as well as bilateral basic hypergeometric functions.
In 2017, Akashet.al.[5] established results involving three bilateral basic hypergeometric functions and ratio of infinite products by using suitable identities and changing the parameters in well known results.

2. Notations

Assuming that |q| <1 , where q is non – zero complex number, this condition ensures that all the infinite product will converge.

$$(a;q)_n = \begin{cases} 1 & ;\ n=0 \\ (1-a)(1-aq)\dots(1-aq^{n-1}) & ;\ n=1,2\dots\dots\dots\dots \end{cases} \quad (2.1)$$

Accordingly , we have –

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n) \quad (2.2)$$

Also ,

$$(a_1,a_2,a_3\dots\dots\dots.a_r;q)_n = (a_1;q)_n,(a_2;q)_n,(a_3;q)_n,\dots\dots\dots,(a_r;q)_n \quad (2.3)$$

Some identities involving q-shifted factorials are

$$(a;q)_{n+k} = (a;q)_n(aq^n;q)_k \quad (2.4)$$

$$(a;q)_{n-k} = \frac{(a;q)_n}{(q^{1-n}/a;q)_k}\left(\frac{-q}{a}\right)^k q^{\binom{k}{2}-nk} \quad (2.5)$$

Where n is a non-negative integer.

Some standard summation and transformation formulae for the basic hypergeometric series which are used in this paper

Heine's transformation of ${}_2\phi_1$series given by Gasper and Rehman[1]

$$\sum_{n=0}^{\infty}\frac{(a,b;q)_n}{(q,c;q)_n}z^n = \frac{(c/b,\ bz;q)_\infty}{(c,\ z;q)_\infty}\sum_{n=0}^{\infty}\frac{(abz/c,\ b;q)_n}{(q,\ bz;q)_n}(c/b)^n \quad (2.6)$$

Slater [3] gave the following result

$${}_1\phi_0(a,\ldots;q;z)=\frac{(az;q)_\infty}{(z;q)_\infty} \qquad |z|<1, |q|<1 \qquad (2.7)$$

q – analog of Euler's Transformation of ${}_2\phi_1$ seriesgiven by Gasper and Rehman[1]

$$\sum_{n=0}^{\infty}\frac{(a,b;q)_n}{(q,c;q)_n}z^n = \frac{(abz/c\,;\,q)_\infty}{(z\,;\,q)_\infty}\sum_{n=0}^{\infty}\frac{(c/a,\ c/b;q)_n}{(q,\ c;q)_n}(abz/c)^n \quad (2.8)$$

q – Gauss summation formula given by Gasper and Rehman[1]

$$\sum_{n=0}^{\infty}\frac{(a,b;q)_n}{(q,c;q)_n}(c/ab)^n = \frac{(c/a,\ c/b;q)_\infty}{(c,\ c/ab;q)_\infty} \qquad (2.9)$$

3. Main Results.

$${}_1\psi_1\begin{bmatrix} b\,;q\,;\,z/q \\ c \end{bmatrix} = \frac{(c/b,\ bz\,,\ q;q)_\infty}{(c,\ z\,,\ cq/bz;q)_\infty}\ {}_1\psi_1\begin{bmatrix} b\,;q\,;\,z/q \\ bz \end{bmatrix} \qquad (3.1)$$

$${}_1\psi_1\begin{bmatrix} b\,;q\,;\,az/q \\ c \end{bmatrix} = \frac{(abz/c\,,\ q/a;q)_\infty}{(q,\ z;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(b;q)_n}{(c;q)_n}(az/q)^n\ {}_2\phi_1\begin{bmatrix} c/b\,,aq^{1-n}/c\ ;q\,;\,q/a \\ q^{1-n}/b \end{bmatrix} \quad (3.2)$$

$${}_2\psi_2\begin{bmatrix} aA\,,b\,;q\,;z \\ B\,,c \end{bmatrix} = \frac{(B/aA\,,\ cq/Aabz\,,c/b\,,bz;q)_\infty}{(q/aA,\ cB/aAbZ\,,\ c\,,\ z;q)_\infty}\ {}_2\psi_2\begin{bmatrix} \frac{Aabz}{c},b\ ;q\,;\,c/b \\ B,bz \end{bmatrix} \qquad (3.3)$$

4. Proof of Main Results.

1. In (2.6) , changing b to bq^n and c to $cq^n$and then using (2.4) we get

$$\sum_{k=0}^{\infty}\frac{(a;q)_k\,(b;q)_{n+k}}{(q;q)_k\,(c;q)_{n+k}}(z)^k = \frac{(c/b,\ bz;q)_\infty}{(c,\ z;q)_\infty}\sum_{k=0}^{\infty}\frac{(abz/c;q)_k(b;q)_{n+k}}{(q;q)_k(bz;q)_{n+k}}(c/b)^k \qquad (4.1.1)$$

Multiply throughout by $(z/q)^n$ and summing over all integers from -∞ to +∞ we get

$$\sum_{n=-\infty}^{\infty}\left(\frac{z}{q}\right)^n\sum_{k=0}^{\infty}\frac{(a;q)_k\,(b;q)_{n+k}}{(q;q)_k\,(c;q)_{n+k}}(z)^k = \frac{(c/b,\ bz;q)_\infty}{(c,\ z;q)_\infty}\sum_{n=-\infty}^{\infty}\left(\frac{z}{q}\right)^n\sum_{k=0}^{\infty}\frac{(abz/c;q)_k(b;q)_{n+k}}{(q;q)_k(bz;q)_{n+k}}(c/b)^k \qquad (4.1.2)$$

Shifting the inner indexn→n-k in (4.1.2) and using (2.7) ,(3.1) is obtained

$$\frac{(aq;q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(b;q)_n}{(c;q)_n}(z/q)^n = \frac{(c/b,\ bz;q)_\infty}{(c,\ z;q)_\infty}\frac{(aq;q)_\infty}{(cq/bz;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(b;q)_n}{(bz;q)_n}(z/q)^n$$

$${}_1\psi_1\begin{bmatrix} b\,;q\,;\,z/q \\ c \end{bmatrix} = \frac{(c/b,\ bz\,,\ q;q)_\infty}{(c,\ z\,,\ cq/bz;q)_\infty}\ {}_1\psi_1\begin{bmatrix} b\,;q\,;\,z/q \\ bz \end{bmatrix}$$

2. In (2.8), changing b to bq^n and c to $cq^n$and then using (2.4) we get

$$\sum_{k=0}^{\infty}\frac{(a;q)_k(b;q)_{n+k}}{(q;q)_k(c;q)_{n+k}}(z)^k = \frac{(b;q)_n}{(\frac{c}{a};q)_n}\frac{(abz/c;q)_\infty}{(z;q)_\infty}\sum_{k=0}^{\infty}\frac{(c/a;q)_{n+k}(c/b;q)_k}{(q;q)_k(c;q)_{n+k}}(abz/c)^k \qquad (4.2.1)$$

Multiply throughout by $(az/q)^n$ and summing over all integers from -∞ to +∞ we get

$$\sum_{n=-\infty}^{\infty} (az/q)^n \sum_{k=0}^{\infty} \frac{(a;q)_k (b;q)_{n+k}}{(q;q)_k (c;q)_{n+k}} (z)^k$$

$$= \frac{(abz/c;q)_\infty}{(z;q)_\infty} \sum_{n=-\infty}^{\infty} (az/q)^n \frac{(b;q)_n}{(c/a;q)_n} \sum_{k=0}^{\infty} \frac{(c/a;q)_{n+k}(c/b;q)_k}{(q;q)_k (c;q)_{n+k}} (abz/c)^k$$

(4.2.2)

Shifting the inner index n→ n-k in (4.2.2) and using (2.5) and (2.6) , (3.2) is obtained

$$\frac{(q;q)_\infty}{(q/a;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(b;q)_n}{(c;q)_n} (az/q)^n$$

$$= \frac{(abz/c;q)_\infty}{(z;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(b;q)_n}{(c;q)_n} (az/q)^n \sum_{k=0}^{\infty} \frac{(aq^{1-n}/c;q)_k (c/b;q)_k}{(q;q)_k (q^{1-n}/b;q)_k} (q/a)^k$$

$${}_1\psi_1 \left[\begin{matrix} b\,;q\,;\, az/q \\ c \end{matrix}\right] = \frac{(abz/c\,,\, q/a;q)_\infty}{(q,\ z;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(b;q)_n}{(c;q)_n} (az/q)^n \ {}_2\phi_1 \left[\begin{matrix} c/b\,, aq^{1-n}/c\ ;q\ ;\ q/a \\ q^{1-n}/b \end{matrix}\right]$$

3. In (2.6), changing b to bq^n and c to $cq^n$and then using (2.4) we get

$$\sum_{k=0}^{\infty} \frac{(a;q)_k (b;q)_{n+k}}{(q;q)_k (c;q)_{n+k}} (z)^k = \frac{(c/b,\ bz;q)_\infty}{(c,\ z;q)_\infty} \sum_{k=0}^{\infty} \frac{(abz/c;q)_k (b;q)_{n+k}}{(q;q)_k (bz;q)_{n+k}} (c/b)^k \qquad (4.3.1)$$

Multiplying throught by $\frac{(A)_n}{(B)_n} (az)^n$ and summing over all integers from -∞ to +∞ we get

$$\sum_{n=-\infty}^{\infty} \frac{(A;q)_n}{(B;q)_n} (az)^n \sum_{k=0}^{\infty} \frac{(a;q)_k (b;q)_{n+k}}{(q;q)_k (c;q)_{n+k}} (z)^k$$

$$= \frac{(c/b,\ bz;q)_\infty}{(c,\ z;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(A;q)_n}{(B;q)_n} (az)^n \sum_{k=0}^{\infty} \frac{(abz/c;q)_k (b;q)_{n+k}}{(q;q)_k (bz;q)_{n+k}} (c/b)^k \qquad (4.3.2)$$

Shifting the inner index n→n-k in (4.3.2) and using (2.5) and (2.9) , (3.3) is obtained

$$\sum_{n=-\infty}^{\infty} \frac{(A;q)_n (b;q)_n}{(B;q)_n (c;q)_n} (az)^n \frac{(q^{1-n}/aA,\ B/A;q)_\infty}{(q^{1-n}/A,\ B/aA;q)_\infty}$$

$$= \frac{(c/b,\ bz;q)_\infty}{(c,\ z;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(A;q)_n (b;q)_n}{(B;q)_n (bz;q)_n} (az)^n \frac{(cq^{1-n}/aAbz\,, B/A;q)_\infty}{(q^{1-n}/A,\ cB/aAbz;q)_\infty}$$

$$\frac{(q/aA,\ B/A;q)_\infty}{(q/A\,,\ \ B/aA;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(aA;q)_n (b;q)_n}{(B;q)_n (c;q)_n} (z)^n$$

$$= \frac{(c/b,\ bz;q)_\infty}{(c,\ z;q)_\infty} \frac{(cq/aAbz\,, B/A;q)_\infty}{(q/A,\ cB/aAbZ;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(Aabz/c\,,\ b;q)_n}{(B\,,\ bz;q)_n} \,.\, (c/b)^n$$

$$\sum_{n=-\infty}^{\infty} \frac{(aA\,, b;q)_n}{(B,c;q)_n} (z)^n = \frac{(B/aA\,,\ cq/Aabz;q)_\infty}{(q/aA,\ cB/aAbZ;q)_\infty} \frac{(c/b,\ bz;q)_\infty}{(c,\ z;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(Aabz/c\ ,\ b;q)_n}{(B\ ,\ bz;q)_n} \,.\, (c/b)^n$$

$${}_2\psi_2 \left[\begin{matrix} aA\,, b\,;q\,;z \\ B\,, c \end{matrix}\right] = \frac{(B/aA\,,\ cq/Aabz\,, c/b\,, bz;q)_\infty}{(q/aA,\ cB/aAbZ\,,\ c\,,\ z;q)_\infty} \ {}_2\psi_2 \left[\begin{matrix} \frac{Aabz}{c}, b \\ B, bz \end{matrix} ;q;\ c/b\right]$$

5. Special Cases

i. Replacing z by c/ab in (3.1) we obtain

$$ {}_1\psi_1\left[\begin{matrix} b\,;q\,;\,c/abq \\ c \end{matrix}\right] = \frac{(\frac{c}{b},\frac{c}{a},q;q)_\infty}{(c,\frac{c}{ab},\frac{q}{a};q)_\infty}\ {}_1\psi_1\left[\begin{matrix} b\,;q\,;\,c/abq \\ c/a \end{matrix}\right] $$

ii. Replacing z by c/b in (3.2) we obtain

$$ {}_1\psi_1\left[\begin{matrix} b\,;q\,;\,ca/bq \\ c \end{matrix}\right] = \frac{(a\,,q/a;q)_\infty}{(q,\ c/b;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(b;q)_n}{(c;q)_n}(ac/bq)^n\ {}_2\phi_1\left[\begin{matrix} c/b\,,aq^{1-n}/c\ ;q\,;\,q/a \\ q^{1-n}/b \end{matrix}\right] $$

iii. Replacing z by c/ab in (3.3) we obtain

$$ {}_2\psi_2\left[\begin{matrix} aA\,,b\,;q\,;c/ab \\ B\,,c \end{matrix}\right] = \frac{(B/aA\,,\,q/A\,,\ c/b\,,\,c/a;q)_\infty}{(q/aA,\,B/A\,,\ c\,,\frac{c}{ab};q)_\infty}\ {}_2\psi_2\left[\begin{matrix} A\,,b\,;q\,;c/b \\ B\,,c/a \end{matrix}\right] $$

6. Reference

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