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RESTRICTED THREE BODY PROBLEM AND SITNIKOV MOTION

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***Abstract*— This paper deals and established the equation of motion of restricted three body problem. We assume the massive bodies having spherical symmetry move about the center of mass in circular orbits. The restricted problem of the three bodies is to describe the motion of the third body. Next we find stationery solution and discussed about stability and solve the Sitnikove problem.**

***Keywords*—** Restricted three body problem, Stationary solution, Curve zero velocity, Stability, Sitnikov problems.

I. INTRODUCTION

Let two massive bodies having spherical symmetry move about their centre of mass in circular orbits. A third mass, the infinitesimal one, moves under the combined gravitational attraction of the two masses but does not influence their motion. The restricted problem of the three bodies is to describe the motion of the third body. If we ignore the presence of the sun, the lack of sphericity of the earth, and the eccentricity of the moon's orbit, the earth-moon system together with a small artificial satellite constitutes such a system of masses.Euler was the first to contribute towards the restricted problem in 1772 in connection with his Lunar Theories. His main contribution was the introduction of a synodic (rotating) co-ordinates system resulting in what is called the Jacobi integral which was discovered by Jacobi (1836). Implications of this integral are numerous. It determines the regions of motion. Its application to celestial mechanics was first made by Hill (1878), Poincare and Birkhoff are the pioneers in the qualitative methods of dynamics.

Poincare’s famous work in three volumes 'Methods Nouvells' completed in (1899) was so new and original that many of its implications are still not clear. We give some important results in regard to the restricted problem. To cope with the situation of demerits of three body problem, we consider one of the masses is so small that gravitational effect due to this mass on other two masses is neglected. This small body is known as ‘infinitesimal body’ and other two are called finite masses.So, in the restricted three body problem the motion of the infinitesimal mass is evaluated in the gravitational field of two finite masses. The earth, moon and artificial satellite system constitute a good example of restricted three body problem.

II. EQUATION OF MOTION

Let us consider a co-ordinate system (O, XYZ) rotating relative to the inertial frame of reference with a constant angular velocity ω about z-axis. Without loss of generality, we can choose the co-ordinate system in such a way that x-axis lies along the line joining the two finite masses m_1 and m_2 with O as barycenter. The motion of m_1 and m_2 are known. We are only to find the motion of m (infinitesimal mass). Let co-ordinate of *m*be (x, y, z). Radius vector from m to m_1 and $m_2$be $\vec{p}_1$ and $\vec{p}_2$ respectively.Kinetic energy of m in rotating frame of reference (O, XYZ) about z-axis is given by

$$T = \frac{1}{2} m\left[(\dot{x} - \omega y)^2 + (\dot{y} + \omega x)^2 + \dot{z}^2\right] \qquad ...(1)$$

Potential energy function V is given by

$$V = -Km\left(\frac{m_1}{p_1} + \frac{m_2}{p_2}\right) \qquad ...(2)$$

Lagrangian (L) = T – V.Equation of motion will be

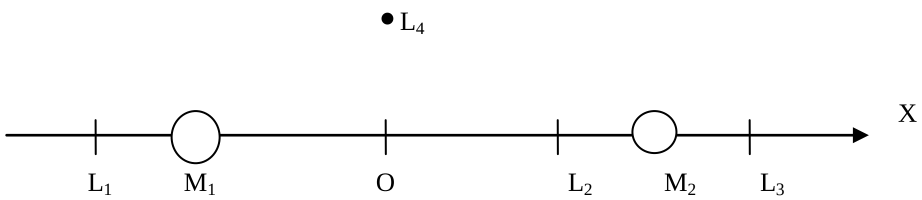
$$\left.\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0 \end{aligned}\right\} \qquad ...(3)$$

whichgives

$$\left.\begin{aligned} \ddot{x} - 2\omega\dot{y} = -\frac{1}{m}\frac{\partial U}{\partial x} \\ \ddot{y} + 2\omega\dot{x} = -\frac{1}{m}\frac{\partial U}{\partial y} \\ \ddot{z} = -\frac{1}{m}\frac{\partial U}{\partial z} \end{aligned}\right\} \qquad ...(4)$$

where $U = -Km\left(\frac{m_1}{p_1} + \frac{m_2}{p_2}\right) - \frac{1}{2} m\omega^2\left(x^2 + y^2\right)$ ...(5)

(i). Stationary Solutions: In 1772 Lagrange's discovered two special solutions of the three – body problem which may be designated stationary solutions. By stationary solution, we mean one in which the geometric configuration of the three masses remains invariant with respect to time. The first solution is known as Straight line solution and the second solution is, the equilateral triangle solution, valid for any masses moving in coplanar circular orbits around their centre mass, with constant angular velocity. Since, in the conservative system of forces, the force function is the function of position co-ordinates. So, if the force components be equated to zero, the solution will give equilibrium points. So, by taking $\frac{\partial U}{\partial x} = \frac{\partial U}{\partial y} = \frac{\partial U}{\partial z} = 0$, we get two types of equilibrium points, they are collinear and triangular respectively. L_1, L_2 and L_3 are collinear equilibrium points in classical restricted three body problem whereas L_4 and L_5 are triangular equilibrium points. The following figure will show their positions.



• L_5

Figure 1.3: Position of Equilibrium Points in Restricted Problem of Three Bodies

(ii). Jacobi Integral: The problem has a well-known Jacobi integral $\upsilon^2 = 2\Omega - c$, where υ is the speed of the infinitesimal mass and $\Omega = \frac{1}{2}\left(x^2 + y^2\right) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$.

(iii). Curves of Zero Velocity

(a). Curves of zero velocities is given by $2\Omega - c = 0$,

(b). When the infinitesimal mass moves in the vicinity of either the first primary or the second primary, it cannot escape and is said to have the 'Hill Stability'.

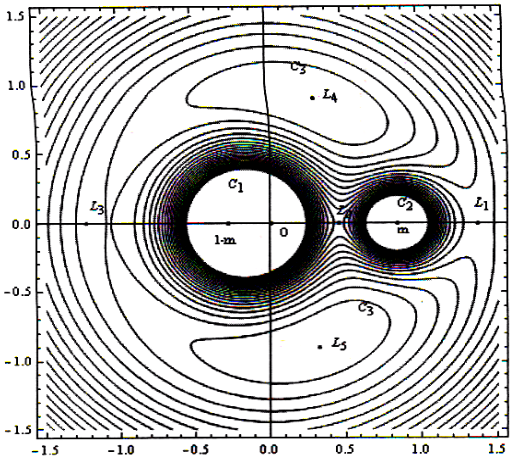


Figure : Contours of Zero Velocity in Restricted Three body Problem

(iv). Sundman Inequality: We know in the inequality

$$c^2 + \left(\frac{dI}{dt}\right)^2 \le 4I\left(U + h\right),$$

(a). for $h \geq 0$, the inequality implies $IU^2 \geq -ch^2$,

(b). for $h < 0$, the inequality implies successively

$c^2 < 4I(U + h)$,

$c^2 h > 4Ih\,(U + h)$,

$IU^2 + c^2 h > I(U^2 + 4Uh + 4h^2) = I(U + 2h)^2 > 0$.

These Inequalities divide the zone of possible motion into three disconnected parts. Such disconnection never happens whenever $n > 3$ this is the major differences between three body problem and more than three body problems.

Stability: Hagihara (1957) considered stability a "fascinating and difficult problem of human culture". He formulated the problem of stability of the solar system as follows: "Will the present configuration of the solar system be preserved for some of the interval of time? Willthe planets eventually fall into the Sun or will some of the planets recede gradually from the sun so that they no longer belong to the solar system? What is the interval of time, at the end of which the solar system deviates from the configuration by apreviously assigned small amount?" Then Hagihara continues: 'The question has long been an acute problem in celestial mechanics since Laplace, not to the Egyptian or the Chaldean civilization. Various mathematicians have often discussed the term “stability” and the solution of the problem becomes more and more complicated and difficult to answer as we dig deeper and deeper into it.

Present day mathematics hardly enables us to answer this question in a satisfactory manner for actual solar system. We must limit ourselves here to describing the present status of the efforts towards solving this fascinating but difficult problem of human culture. The phase space of an autonomous Hamiltonian system of three degrees of freedom is 6-dimensional, but if we consider only orbits with the same energy the phase space becomes 5-dimensional. A Poincare surface section is thus 4-dimensional. If the potential is symmetric with respect to the plane *z* = 0, this plane is appropriate surface of section $(x, y, \dot{x}, \dot{y})$. The *z*-velocity, $\dot{z}$ is then found from the energy equation

$$H = H(x, y, z, \dot{x}, \dot{y}, \dot{z}) = h \qquad \text{...(6)}$$

For $z = 0$. If we start an orbit at a point $(x_0, y_0, \dot{x}_0, \dot{y}_0)$ with $z = 0$ and fixed *h* we find the next intersection with the surface of section by solving the Hamiltonian equations of motion. If we denote now the variable $(x, y, \dot{x}, \dot{y})$ by $X(x_1, x_2, x_3, x_4)$, we find the first consequent of $X_0$by a transformation of the form $X_1 = TX_0$

…..(7)

Where T represents four functions $f_t = f_i(x_{10}, x_{20}, x_{30}, x_{40})$.

A small change $\zeta_0(\zeta_{10}, \zeta_{20}, \zeta_{30}, \zeta_{40})$ initial condition X_0gives also small variations of x_1 which in the linear approximation are

$$\zeta_{i1} = \sum_{i}^{4} \alpha_{ij} \zeta_{j_0} \qquad \text{……. (8)}$$

Or in matrix form

$$\overline{\zeta} = A\overline{\zeta}_0$$

The coefficient $\alpha_{ij}$are the partial derivatives of $f_i$with respect to x_{j0}. These coefficient are found by calculating 4 orbits near the original one with deviations (σ, 0, 0, 0), (0, σ, 0, 0), (0, 0, σ,0) and (0, 0, 0, σ), respectively, for a small but arbitrary σ.

If there is a periodic orbit close to X_0we can find it by Newton method [Magnenat 1982b]. Namely if we set $\zeta_{i1} = \zeta_{i0}$ in (7) we solve this system for ζ_{i0} and find a better approximation $X_0^{'} = X_0 + \zeta_0$ for the periodic orbit. By repeating this procedure a few times we find the periodic orbit with the required accuracy.

The variables (x_1, x_2, x_3, x_4)on the surface section are not necessarily canonical, but they can be expressed in terms of canonical variables. As a consequence [Hadjidemetriou 1975] the volumes in the 4-D surface of section are preserved and this implies that the determinant of the matrix *A* is equal to 1. i.e. *det(A)* = 1 Furthermore, the eigenvalues of the matrix *A* are inverse in pairs, because the system is derived from a Hamiltonian. This means that the characteristic equation

$$|A - \lambda I| = 0,$$

where*I* is the unit matrix, is of the form

$$\lambda^4 + \alpha\lambda^3 + \beta\lambda^2 + \alpha\lambda + 1 = 0, \qquad \text{…………………….. (9)}$$

where $\alpha = -Trace(A) = -(a_{11} + a_{22} + a_{33} + a_{44})$

and $\beta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{14} \\ a_{41} & a_{44} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{24} \\ a_{42} & a_{44} \end{vmatrix} + \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix}.$

Equation (37) can be written as

$$(\lambda^2 + b_1\lambda + 1)(\lambda^2 + b_2\lambda + 1) = 0$$

here, $b_1 = \frac{1}{2}\left(\alpha + \sqrt{\Delta}\right), b_2 = \frac{1}{2}\left(\alpha - \sqrt{\Delta}\right)$

with $\Delta = \alpha^2 - 4(\beta - 2)$

Thus the eigenvlues are

$$\lambda_1, \lambda_2 = \frac{1}{2}\left(-b_1 \pm \sqrt{b_1^2 - 4}\right),$$
$$\lambda_3, \lambda_4 = \frac{1}{2}\left(-b_2 \pm \sqrt{b_2^2 - 4}\right), \qquad \text{…………..(10)}$$

It is obvious that $\lambda_1\lambda_2 = 1, \ \lambda_3\lambda_4 = 1$ i.e. the eigenvalues are inverse in pairs.

In the particular case when the dynamical system is 2-Dimensional, the Poincare surface section is 2-Dimensional and there is only one stability parameter *b*. If *a* is the Henon parameter in the case of symmetric orbits. Thus *b* = –2*a*.

We have the following four type of stability, using the terminology of Contopoulos and Magnenat [1985].

(1) Stable orbits: Then the eigenvalues are complex conjugate $\left(\lambda_2 = \bar{\lambda}_1,\ \lambda_4 = \bar{\lambda}_3\right)$ on the unit circle. This happens if b_i are real $(i.e.\ \Delta > 0)$ and $|b_1| < 2,\ |b_2| < 2$. In this case $\lambda_i = e^{i\phi_i}$,where $\phi_2 = -\phi_1, \phi_4 = -\phi_3$ and $b_1 = -2Cos\phi_1,\ b_2 = -2Cos\phi_2$.

(2) Simply unstable orbit: Then two eigenvalues are on the unit circle and two on the real axis (positive or negative). Thus $\Delta > 0$ and $|b_1| < 2,\ |b_2| > 2$ or $|b_1| > 2,\ |b_2| < 2$.

(3) Doubly unstable orbits: Then all the four eigenvalues are on the real axis. Thus $\Delta > 0$ and $|b_1| > 2,\ |b_2| > 2$.

(4) Complete unstable orbits: Then all four eighenvalues are complex, but outside the unit circle. Two of them are outside and two of them inside the unit circle. In this case $\Delta > 0$.

Broucke [1969] used a more detailed classification into seven types:

(l) Stability, (2) Complex Instability, (3) Even-Odd Instability, (4) Even-Even Instability, (5) Odd-Odd Instability, (6) Even Semi-Instability, (7) Odd Semi-Instability.

Sitnikov problem: The Sitnikov problem [1961] is a special case of restricted three body problem. It refers to the motion of the test particle along an axis perpendicular to the plane of motion of two equal primaries that move on elliptic orbits. The axis passes through the center of mass of the system.

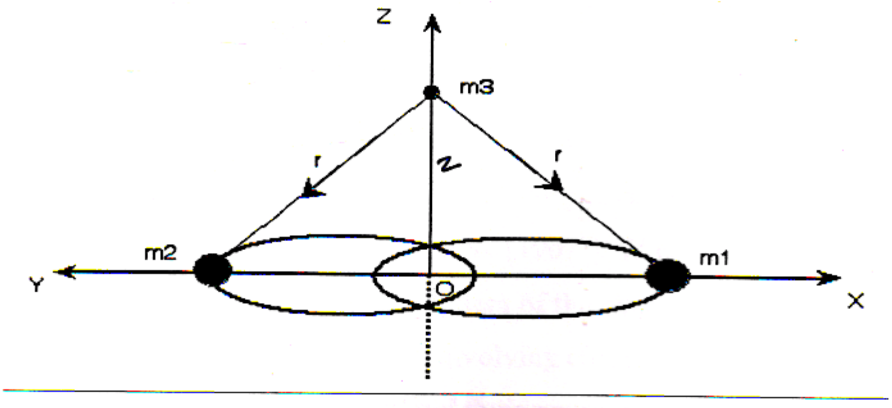


Fig6: The Sitnikov problem.

The polar co ordinates of the one primary around the center of mass are (r, v) and the elliptic orbit is given by the relation

$$r = \frac{1-e^2}{2(1+eCosv)} \qquad \text{…………….(11)}$$

According to Kepler's first law, where e is the eccentricity, v the true anomaly and the semi major axis is equal to $\frac{1}{2}$.

Then the test particle moves along the Z-direction according the law

$$\frac{d^2z}{dt^2} = -\frac{z}{\left(r^2+z^2\right)^{\frac{2}{3}}}, \qquad \text{………………….(12)}$$

The orbit of the test particle may be periodic, quasi periodic, chaotic, or escaping to infinity. The strangest cases of chaotic orbits are orbits that are unbounded yet nonescaping. Such orbits go to arbitrarily large distances along the Z-axis, but they always return near the origin.

If the eccentricity *e* is zero the problem is integrable and chaotic orbit does not exist. But if *e* is different from zero, even if it is very small, the whole complexity of the non-integrable problems appears [Liu and Sun, 1990].

The Simplest version of Sitnikov problem is known as circular Sitnikov or MacMillan problem in which the eccentricity *e* of the primaries is equal to zero. This dynamical model was first described by Pavanini [1907)]. W.D MacMillan [1910] presented it as an example of an integrable system of the restricted three-body problem and gave a solution in terms of a quadrature involving elliptic integrals, where the primaries are moving in circular orbits around their centre of mass and discussed in detail and he showed the integrability of the equation of motion with the aid of elliptic integrals. He found the solution as

$$z = a\left[Sin\tau + \left\{\frac{3}{64}Sin\tau + \frac{3}{64}Sin3\tau\right\}\mu + \left\{\frac{79}{4096}Sin\tau + \frac{27}{1024}Sin3\tau + \frac{29}{4096}Sin5\tau\right\} + \mu^2 + ...\right] \qquad \text{………………….(13)}$$

III. CONCLUSIONS

We established the equation of motion of restricted three body problem when two massive bodies having spherical symmetry more about their centre of mass in circular orbits. Next we find stationary solution and we get, two type of equilibrium points, they are collinear and triangular respectively. L_1, L_2 and L_3 are collinear equilibrium points in classical restricted three body problem whereas L_4 and L_5 are triangular equilibrium points. When the infinitesimal mass moves in the vicinity of either the first primary or the second primary, it cannot escape and is said to have the 'Hill Stability' and solve the Sitnikov problems.

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