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NEIGHBOURHOOD CONTRACTION OF SOME IRREGULAR GRAPHS

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Abstract

A connected graph G(V,E) is said to be a neighbourly irregular graph, if no two adjacent vertices of G have the same degree. G is said to be a highly irregular graph if for every vertex of G no two neighbours are of same degree. G is said to be neighbourhood highly irregular, for any vertex v, if any two distinct vertices in the open neighbourhood of v, have distinct closed neighbourhood sets. The neighbourhood contracted graph G_v of G, with respect to the vertex v in G is the graph with the vertex set V-N(v), where two vertices u, w are adjacent in G_v, if either w = v and u is adjacent to any vertex of N(v) in G or u, w $\notin$ N[v] and u, w are adjacent in G. In this paper, we discuss the conditions needed for both G and G_v to be irregular graphs of same kind.

Keywords: Neighbourhood contraction, irregular graphs, neighbourly irregular graphs.

AMS Subject Classification Code(2010): 05 C (Primary)

1 Introduction

The graphs taken under consideration are finite, simple, undirected and connected. One can refer [3] for further notations and terminology. In a graph *G(V, E)*, for any vertex $v \in V$, the *open neighbourhood*, denoted by *N(v)* of *v* is the set of all vertices adjacent to *v*. That is, $N(v) = \{u \in V(G) / uv \in E(G)\}$. The *closed neighbourhood* of *v* is defined by $N[v] = N(v) \cup \{v\}$. For any subset *S* of *V*, $N(S) = \bigcup_{v \in S} N(v)$. In general, for any positive integer *k*, let $N_k(v) = \{u \in V(G) / d(u,v) = k\}$.

A full vertex of *G* is a vertex which is adjacent to every other vertices of *G*. A graph *G* is said to be an *r – regular graph*, if degree of every vertex in *G* is of same degree *r*. For any subset *S* of *V*, a vertex $v \in S$ is said to be an *enclave* of *S*, if $N[v] \subseteq S$. In particular, *v* is a *complete enclave* if $N[v] = S$.

The *distance d(u,v)* [5], between any two vertices *u* and *v* is the length of a shortest path between them. The *eccentricity e(u)* of a vertex *u* is the distance of a farthest vertex from *u*. The *radius, rad(G)* of *G*, is the minimum eccentricity and the *diameter, diam(G)* of *G*, is the maximum eccentricity of the graph *G*. The *Girth* of a graph *G* is defined as the length of the smallest cycle in *G*.

For any vertex $v \in V(G)$, the *neighbourhood Contracted graph* G_v of *G*, introduced by S.S. Kamath and Prameela Kolake in [8], is a graph with the vertex set $V - N(v)$, where two

vertices $u, w \in V - N(v)$ are adjacent in G_v either if $w=v$ and u is adjacent to any vertex of $N(v)$ in G or if $u,w \notin N[v]$ and u, w are adjacent in G. For example, a graph G and its neighbourhood contracted graph G_v are shown in Figure 1. Some interesting results on neighbourhood contraction of graphs can be found in [4] and [8].



G ***G_v***

Figure 1

Irregularity in graphs is an interesting branch of research in graph literature. A variety of irregular graphs have been defined and studied for their properties. A connected graph G is said to be a *highly irregular graph* or simply a HI graph[2], if for every vertex of G no two neighbours are of same degree. Some highly irregular graphs are shown in Figure 2.

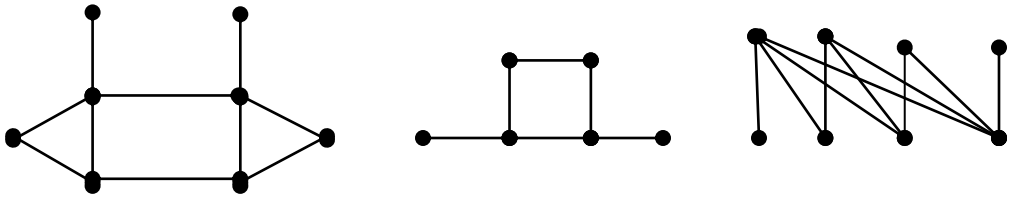


Figure 2

A connected graph G is said to be a *totally segregated graph[7]*, if no two adjacent vertices of G have the same degree. Later it has been independently as neighbourly irregular graphs in [6]. Some neighbourly irregular graphs are shown in Figure 3.

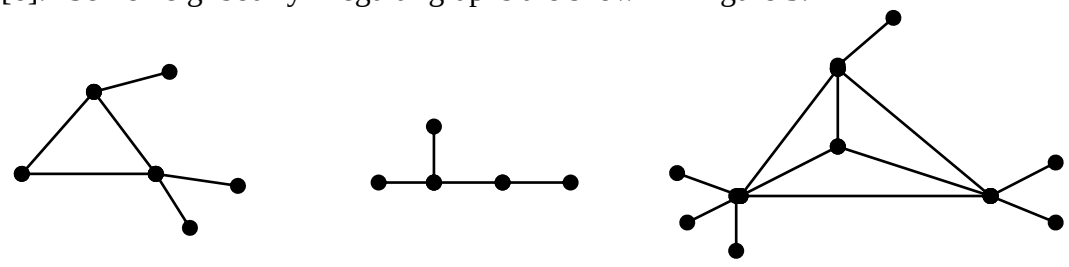


Figure 3

A connected graph G is said to be *neighbourhood highly irregular[10]* (or simply NHI), if, for any vertex v, any two distinct vertices in the open neighbourhood of v, have distinct closed

neighbourhood sets, that is, if for any vertex v, the vertices u, $w \in N(v)$, $(u \neq w)$, implies that $N[u] \neq N[w]$.

Examples of NHI graphs are given in Figure 4.

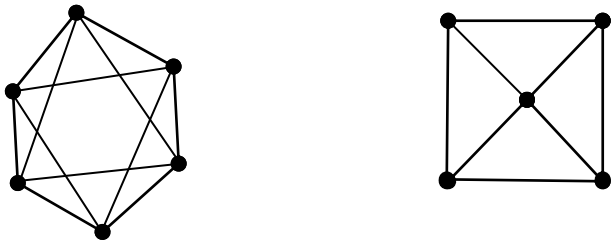


Figure 4

This paper concentrates on neighbourhood contraction in some irregular graphs.

2 Main Results

In this section, we discuss the various conditions under which G_v is an NI graph. We refer the $(i,j)^{th}$ entry of adjacency matrix A of G as a_{ij} and that of A^2 as b_{ij}. Let $d(w)$ denote the degree of a vertex w in G, whereas, $d_v(w)$ denote its degree in a contracted graph G_v.

To avoid ambiguity, we use the notation $N_k(v) = \{u \in V(G)/d(u,v) = k \text{ in } G\}$ and $N^*_k(w) = \{u \in V(G_v)/d(u,w) = k \text{ in } G_v\}$. One can easily verify that $d_v(v) = |N_2(v)|$. Let V_i denote the set of all vertices of degree i in G. Note that in an NI graph, $<V_i>$ is independent. Any vertex adjacent to a pendant vertex is called the support of the pendant.

Proposition 2.1 In a connected NI graph G, for any pendant vertex v whose support has degree 2, G_v is NI if and only if $G \ncong P_3$.

Proof Let G be a connected NI graph with a pendant vertex v whose support w has degree 2. Let u be a neighbour of w other than v, that is, $N(w) = \{u,v\}$. We note that v is again a pendant vertex in G_v,. Since $G \ncong P_3$, $N^*(v) \subseteq V_i$, for some $i > 1$. Hence the endpoints of the edge uv has distinct degrees in G_v. This forces that G_v is also NI.

Conversely assume that G_v is NI. If $G \cong P_3$, then $G_v \cong P_2$, which is not NI, a contradiction and hence $G \ncong P_3$.. ■

Theorem 2.2 In a connected NI graph G, with $v \in V_1$ and $w \in N(v) \cap V_i$, where $i \neq 2$, G_v is NI if and only if $N(w) \cap V_{d(w)-1} = \phi$.

Proof Let G be any connected NI graph. Let v be a pendant vertex of G with w being its support vertex. The vertex $w \notin V_2$, then $d(w) - 1 \neq 1$. This makes sure that $v \notin N(w) \cap V_{d(w)-1}$. Also G is NI. Hence w is not pendant. We note that $N^*(v) = N(w) \setminus \{v\}$. Hence $d_v(v) = d(w) - 1$. Since v is pendant, $|N(u) \cap N(v)| = 1$.

Suppose G_v is NI. On contrary to our claim, suppose $N(w) \cap V_{d(w)-1} \neq \phi$. And let $u \in N(w) \cap V_{d(w)-1}$..

Now $d_v(u) = d(u) - |N(u) \cap N(v)| + 1$

$= d(u)$

$= d(w) - 1$

$= d_v(v)$, a contradiction to the fact that G_v is NI.

Therefore $N(w) \cap V_{d(w)-1} = \phi$.

Conversely, when $N(w) \cap V_{d(w)-1} = \phi$ in G, no neighbour of w is of degree $d(w) - 1$ in G. Now consider G_v. Let $u \in N^*(v)$ implies $u \in N(w)$. Hence $d(u) \neq d(w) - 1$.

Then $d_v(u) = d(u) - |N(u) \cap N(v)| + 1$

$= d(u)$

$\neq d(w) - 1$

$= d_v(v)$. Hence G_v is also NI. ■

Proposition 2.3 Let G be an NI tree with a unique vertex v of maximum degree. If $N(v) \cap V_1 = \phi$, then G_v is also an NI tree.

Proof Let G be an NI tree. Let v be the only vertex of maximum degree in G. By hypothesis, v is adjacent to no pendant vertex in G. Then $|N_2(v)|$ is at least Δ. Since G is a tree, every vertex in $N_2(v)$ is adjacent to exactly one vertex in $N(v)$. This ensures that for any vertex $u \in N_2(v)$, $d^*(u) = d(u) < \Delta$. Hence no newly added edge in G_v has end points of same degree. Also since G is NI, no edge common to both G and G_v has same degree for its end vertices. Thus we arrive at a conclusion that G_v is NI. ■

The concept of support of a vertex [1] has been introduced recently. The support of a vertex v is denoted by $s(v)$, and is defined by the sum of degrees of all vertices adjacent to v. That is, $s(v) = \sum_{u \in N(v)} d(u)$.

Theorem 2.4 For any NI tree G, the contracted graph G_v is also an NI tree if and only if $s(v) \neq d(v) + d(w)$, for any vertex $w \in N_2(v)$.

Proof Let G be any NI tree. Assume that G_v is also NI. On contrary, suppose $s(v) = d(v) + d(w)$, for some vertex $w \in N_2(v)$. Since G is a tree, no two vertices of $N(v)$ are adjacent. In addition, no two vertices of $N(v)$ have a common neighbour other than v in G. Combining these two facts, we get $s(v) = |N_2(v)| + d(v)$.

Also we know that $d_v(v) = |N_2(v)|$. Therefore $s(v) = d_v(v) + d(v)$. By hypothesis, we get $d(w) = d_v(v)$. We note that $d_v(w) = d(w) - |N(w) \cap N(v)| + 1$. But since G is a tree, $|N(w) \cap N(v)| = 1$ and hence $d_v(w) = d(w)$. This implies that $d_v(w) = d_v(v)$ which is a contradiction since G_v is NI. Hence $s(v) \neq d(v) + d(w)$, for any vertex $w \in N_2(v)$.

Conversely suppose $s(v) \neq d(v) + d(w)$, for any vertex $w \in N_2(v)$. Again by the same argument, we get $d_v(v) = |N_2(v)| = s(v) - d(v) \neq d(w) = d_v(w)$ for all $w \in N_2(v)$. This ensures that

the edges incident with v in G_v have end vertices of different degrees. Given that G is NI. And so, all other edges also do not have end vertices of same degree. Hence G_v is NI. ■

Lemma 2.5 Let G be a connected graph. If there exists a vertex $v_i \in V(G)$ such that $|N_2(v_i)| + b_{ij} = d(v_j)+1$, for some $v_j \in N_2(v_i)$, then G_{v_i} is not an NI graph.

Proof Let G be any connected graph. Let $v_i \in V(G)$ and $v_j \in N_2(v_i)$.

It is quite easy to note that b_{ij} = the number of walks of length 2 from v_i to v_j

$= |N(v_j) \cap N(v_i)|$

Suppose $|N_2(v_i)| + b_{ij} = d(v_j) + 1$, for some $v_j \in N_2(v_i)$. Also $d_{v_i}(v_i) = |N_2(v_i)|$.

Then by the definition of G_{v_i}, $d_{v_i}(v_j) = d(v_j) - |N(v_j) \cap N(v_i)| + 1$

$= d(v_j) - b_{ij} + 1$

$= |N_2(v_i)|$ (hypothesis)

$= d_{v_i}(v_i)$

Hence G_{v_i} is not NI. ■

Lemma 2.6 In any connected graph G, if there are two adjacent vertices v_j and v_k in $N_2(v_i)$, for any vertex $v_i \in V(G)$ such that $d(v_j) - d(v_k) = b_{ij} - b_{ik}$, then G_{v_i} is not an NI graph.

Proof Let G be a connected graph. Let $v_i \in V(G)$. Suppose there are any two adjacent vertices v_j and v_k in $N_2(v_i)$ such that $d(v_j) - d(v_k) = b_{ij} - b_{ik}$.

Now, by the definition of neighbourhood contracted graph,

$d_{v_i}(v_j) = d(v_j) - |N(v_j) \cap N(v_i)| + 1$

$= d(v_j) - b_{ij} + 1$.

Also, $d_{v_i}(v_k) = d(v_k) - |N(v_k) \cap N(v_i)| + 1$

$= d(v_k) - b_{ik} + 1$.

By the hypothesis, $d(v_j) - d(v_k) = b_{ij} - b_{ik}$.

$d(v_j) - b_{ij} + 1 = d(v_k) - b_{ik} + 1$.

$d_{v_i}(v_j) = d_{v_i}(v_k)$

This concludes that G_{v_i} is not NI. ■

Lemma 2.7 In any connected graph, if there exist a vertex v_j in $N_2(v_i)$ such that $d(v_j) - d(w) = b_{ij} - 1$, for any vertex $w \in N(v_j) \backslash N(v_i)$, then G_{v_i} is not an NI graph.

Proof Let *G* be any connected graph. Let $w \in N(v_j) \backslash N(v_i)$. Then $d(w,v_i) = 3$. From the definition of Neighbourhood contracted graph G_v, we know that for any vertex *u* such that $u \in N_k(v_i)$, for any k >2, $d(u) = d_v(u)$. Hence $d(w) = d_{v_i}(w)$.

But as in Lemma 2.1, $d_{v_i}(v_j) = d(v_j) - b_{ij}+1$

$= d(w)$ (hypothesis)

$= d_{v_i}(w)$. Hence the lemma. ■

Theorem 2.8 Let *G* be any NI graph with a vertex v_i. Then G_{v_i} is an NI graph if and only if the following conditions hold for any vertex $v_i \in V(G)$. (i) $|N_2(v_i)| + b_{ij} \neq d(v_j)+1$, for all $v_j \in N_2(v_i)$ (ii) $d(v_j) - d(v_k) \neq b_{ij} - b_{ik}$, for any two adjacent vertices $v_j , v_k \in N_2(v_i)$

(iii)$d(v_j) - d(w) \neq b_{ij} - 1$, for any two adjacent vertices $v_j \in N_2(v_i)$ and $w \in N_3(v_i)$.

Proof Let *G* be an NI graph and let v_i be any vertex of *G*. If any of the conditions (i), (ii) and (iii) fails, then by the above three lemmas, G_{v_i} is not NI.

Conversely, assume that G_{v_i} is not an NI graph. Then there exists an edge $v_jv_k \in E(G_{v_i})$ such that $d_{v_i}(v_j)= d_{v_i}(v_k)$. Let $v_j \in N^*_r(v_i)$ and $v_k \in N^*_s(v_i)$ for some integers *r* and *s*. Without loss of generality we can assume that $r \leq s$.

Case 1 If $r \geq 2$ and $s \geq 2$, then $d_{v_i}(v_j) = d_{v_i}(v_k)$ implies $d(v_j)= d(v_k)$. This is a contradiction to the fact that *G* is NI.

Case 2 If $r = 1$ and $s = 2$, then $v_j \in N_2(v_i)$ and $v_k \in N_3(v_i)$. Now $d_{v_i}(v_j) = d_{v_i}(v_k)$ implies $d(v_j) - b_{ij} +1 = d(v_k)$ which contradicts (iii).

Case 3 If $r = s = 1$, then we have $v_j, v_k \in N_2(v_i)$. Here $d_{v_i}(v_j) = d_{v_i}(v_k)$ implies $d(v_j) - b_{ij} +1 = d(v_k) - b_{ik} + 1$ which is a contradiction to (ii).

Case 4 If $r = 0$, then $s = 1$. In this case, $v_j = v_i$ and $v_k \in N_2(v_i)$. Therefore $d_{v_i}(v_j) = d_{v_i}(v_k)$ implies $|N_2(v_i)| = d(v_k) - b_{ik} +1$, again a contradiction to (i).

Hence we conclude that G_{v_i} is an NI graph. ■

Corollary 2.9 Let *G* be an NI graph of diameter at most two. Then G_{v_i} is an NI graph if and only if the following conditions hold for any vertex v_i: (i) $|N_2(v_i)| + b_{ij} \neq d(v_j)+1$, for all $v_j \in N_2(v_i)$
(ii) if $d(v_j) - d(v_k) \neq b_{ij} - b_{ik}$, for any two adjacent vertices $v_j , v_k \in N_2(v_i)$.

Proof The result follows as $|N_3(v_i)| = \phi$. ■

Corollary 2.10 Let *G* be an NI graph of girth at least 6. G_{v_i} is an NI graph if and only if the following conditions hold, for any vertex v_i: (i) $|N_2(v_i)| + b_{ij} \neq d(v_j)+1$, for all $v_j \in N_2(v_i)$ (ii) if $d(v_j) - d(w) \neq b_{ij} - 1$, for any two adjacent vertices $v_j \in N_2(v_i)$ and $w \in N_3(v_i)$.

Proof Since the girth of G is at least 6, there is no edge in $<N(v_i)>$ and $<N_2(v_i)>$ and the result follows. ■

Theorem 2.11 Let G be an NHI graph. Then for any vertex $v \in V(G)$, G_v is an NHI graph if and only if $N_2(v)$ contains no complete enclave.

Proof Let G be an NHI graph and let $v \in V(G)$. Suppose that G_v is NHI. On contrary, let $w \in N_2(v)$ be a complete enclave. Then $N[w] = N_2(v)$. Then in G_v, $N^*[w]=N^*[v]$ which is a contradiction to our assumption that G_v is NHI. Hence $N_2(v)$ has no complete enclave. Conversely, assume that G is an NHI graph in which $N_2(v)$ has no complete enclave. Then G_v has no two vertices with same closed neighbourhood and hence G_v is NHI.

■

Theorem 2.12 Let G be a HI tree. Then for any maximum degree vertex $v \in V(G)$, G_v is HI if and only if $G \cong P_4$.

Proof Consider a HI tree G with a vertex v of maximum degree Δ. By the definition of a HI graph, $N(v)$ contains Δ vertices each of which has distinct degree ranging from 1 to Δ. Now in G_v, v is adjacent to the neighbours of all vertices in $N(v)$.

When $\Delta = 2$, $G \cong P_4$ and hence the corresponding $G_v \cong P_2$ which is trivially HI. If $\Delta = 3$, then in G_v, $d_v(v) = \Delta$ but $N^*(v)$ contains at least two vertices of same degree which forces that G_v is not HI.

Therefore we assume that $\Delta > 3$. Since G is a *HI* tree, no two neighbours of v have a common neighbour. As v is adjacent to a vertex of degree Δ in G, we get $d_v(v) = \Delta(\Delta-1)/2 > \Delta$ and thus v becomes the unique vertex of maximum degree in G_v. This contradicts the property of a HI graph that a vertex of maximum degree must contain another vertex of maximum degree as its neighbour. Hence G_v is not an HI tree. ■

Theorem 2.13 For a HI graph G, if v is a pendant vertex adjacent to a vertex of maximum degree in G, then G_v is not a HI graph.

Proof Let G be a HI graph such that v is a pendant vertex. Also by the hypothesis, v is adjacent to a vertex of maximum degree Δ in G. A HI graph always contains P_4 as an induced subgraph in which internal vertices (say u and w) are of maximum degree in G and the external vertices are pendant in G.

Without loss of generality, let $v \in N(u)$. In G_v, $d_v(v) = \Delta - 1$. We note that $<V_\Delta>$ in any HI graph is disjoint union of copies of K_2. Now $d_v(w) = \Delta$ and w has no neighbour of maximum degree Δ in G_v but it has two neighbours of same degree $\Delta - 1$. Thus G_v is not HI. ■

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