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Existence And Uniqueness Results Of Solution To The Caputo-Type Fractional Differential Equation

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Abstract:In this paper we bring the proof of existence and uniqueness of solution to the existence and uniqueness results of solution to the Caputo-type fractional differential equation problem of linear fractional differential equation and Finally we solve some linear fractional differential equation

Keyword: Fractional Derivatives, Laplace Transform,Convolution of functions, existence and uniqueness of solution.

Introduction: Fractional differential equations are the generalization of ordinary differential equations to arbitrary noninteger orders. The fact, that the fractional derivative(integral) is an operator which includes integer orderderivatives(integrals) as special cases, is the reason why in present fractional differential equations becomesvery popular and many applications are available. The fractional differential equations are of great importancebecause these are more precise in the modeling of many phenomenon, for instance, the nonlinear oscillationsof earthquake can be described by the fractional differential equations. These differential equations are alsovery important to describe the memory and hereditary properties of various materials and phenomenon,this characteristic of fractional differential equations makes the fractional-order models more realistic andpractical than the classical integer-order models. Recent work on fractional differential equations shows anoverwhelming interest in this direction.

2.Overview: Fractional Calculus

Fractional Calculus is a term used for the theory of derivatives and integrals of arbitrary order, which generalize the notion of integer order differentiation and n-fold integration. The idea behind Fractional calculus is to generalize the definition of differentiation and integration with order $n \in \mathbb{N}$ to order $s \in \mathbb{R}$. The first discussion [9] on Fractional Calculus began in 1695 in a letter to L'Hopital by Leibniz in which he discussed about calculus of arbitrary order. Fractional Calculus is three centuries old. Few names that laid the foundation of Fractional Calculus are Abel, Liouville, Riemann, Euler, Caputo etc. Fractional Calculus has recently been applied in various areas of engineering, science, finance, applied mathematics and bio engineering.[10] . It has earlier been observed that derivatives of non-integer order are useful for describing the properties of various real materials like polymer, rocks etc. Also the fractional order models were found more logical to talk an discuss about than the integer-order models. In this paper we are focusing on Fractional Derivatives. Different people gave different definitions for the Fractional Derivative. Few definitions are :

2.1Grunwald-Letnikov Fractional Derivatives: Let us consider a continuous function f(t),We define:

$$ {}_aD_t^p f(t) = \lim_{\substack{h\to 0 \\ nh=t-a}} h^{-p} \sum_{r=0}^{n} (-1)^r \binom{p}{r} f(t-rh) $$

The above formula has been obtained under the assumption that the derivatives $f^{(k)}(t)$ (k=1,2,3,. . . ,m+1) are continuous in the closed interval [a,t] and that m is the integer number satisfying m>p-1.

2.2 Riemann-Liouville Derivatives:

$$ {}_aD_t^p f(t) = (\frac{d}{dt})^{m+1} \int_a^t (t-\tau)^{m-p} f(\tau)d\tau, (m \le p < m+1) $$

2.3 Caputo's Fractional Derivatives: The definition of the fractional differentiation of the Riemann-Liouville Derivatives type played an important role in the development of the theory of fractional derivativesand for its applications in pure mathematics. However, the demands of modern technology require a certain revision of well established mathematical approach .The Caputo approach provides an interpolation between an integer order derivatives:

$$ {}^CD^\alpha f(x) = \frac{1}{\Gamma(\alpha-n)} \int_a^x \frac{f^{(n)}(u)}{(x-u)^{(\propto-n+1)}}, n-1 <\propto< n, \propto\in \mathbb{R}, n \in \mathbb{N} $$

2.4 Euler's Fractional Derivatives:

$$ \frac{d^\propto}{dt^\propto}[t^\beta] = D_t^\alpha[t^\beta] = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha}, \alpha \in \mathbb{R} $$

.

2.5Sequential Fractional Derivatives: The main idea of differentiation and integration of arbitrary order is the generalization of iterated integration and differentiation. In all these approaches we replace the integer valued parameter n of a operator denoted by $\frac{d^n}{dt^n}$ with a non integer parameter p. However,we can assume that the n-th order differentiation is simply a series of n first order differentiation .So, considering more general expressions

$$ D_t^\alpha = D_t^{\alpha_1} D_t^{\alpha_2} D_t^{\alpha_3} \dots\dots\dots . D_t^{\alpha_n} $$

Where $\alpha = \alpha_1 + \alpha_2 + \alpha_3 + \cdots \dots\dots . \alpha_n$,which we will also call the sequential fractional derivatives.

Indeed, Riemann-Liouville Derivatives can be written as

$$ {}_aD_t^p f(t) = \frac{d}{dt}\frac{d}{dt} \dots\dots . \frac{d}{dt}\, {}_aD_t^{-(n-p)} f(t) \quad (\text{n-1} \le p < n) $$

While the Caputo fractional differential operator can be written as

$$ {}^CD^\alpha f(x) = {}_aD_t^{-(n-p)} \frac{d}{dt} \dots\dots . \frac{d}{dt} \text{f(t)} \quad (\text{n-1} < p \le n-1) $$

2.6Properties of Fractional Derivatives:

Fractional Derivatives satisfy almost all the properties that hold for[5] ordinary derivatives. We are aware of the general properties of the derivative operator $D_t^n, n \in \mathbb{N}$. Below mentioned are the properties of Fractional Derivative that can be easily verified:

- $D_t^\propto[f(t)g(t)] = \sum_{k=0}^{\infty}\binom{\propto}{k} D_t^{\propto-k}[f(t)]D_t^k[g(t)]$

  where $\binom{\propto}{k} = \frac{\Gamma(\propto+1)}{\Gamma(k+1)\Gamma(\propto+1-k)}$.

- $D_t^\propto[f(t)C] = \sum_{k=0}^{\infty}\binom{\propto}{k} D_t^{\propto-k}[f(t)]D_t^k[C] = D_t^\propto[f(t)]C$.

- $D_t^\propto[h(t)+g(t)] = \sum_{k=0}^{\infty}\binom{\propto}{k} D_t^{\propto-k}[t^0]D_t^k[h(t)+g(t)] = D_t^\propto[h(t)]+D_t^\propto[g(t)]$.

- $D_t^\propto[h(at)] = a^\propto D_x^\propto[h(x)]\ , x = at$.

- $D_t^\propto[t^{-m}] = (-1)^\propto \frac{\Gamma(m+\propto)}{\Gamma(m)} t^{-(m+\propto)}$.

- $D_t^{\mu+\nu}[f(t)] = D_t^\mu[D_t^\nu(f(t))] = D_t^\nu[D_t^\mu(f(t))]$.

$D_t^{-1}\left[t^\beta\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+1)} t^{\beta+1} = \frac{t^{\beta+1}}{\beta+1}$, where $\alpha \in D_t^\propto[f(t)g(t)] = \sum_{k=0}^{\infty}\binom{\propto}{k} D_t^{\propto-k}[f(t)]D_t^k[g(t)]$,

where $\binom{\propto}{k} = \frac{\Gamma(\propto+1)}{\Gamma(k+1)\Gamma(\propto+1-k)}$.

- $D_t^\propto[f(t)C] = \sum_{k=0}^{\infty}\binom{\propto}{k} D_t^{\propto-k}[f(t)]D_t^k[C] \ = D_t^\propto[f(t)]C$Where$C$ is an arbitrary constant.

- $D_t^\propto[h(t)+g(t)] = \sum_{k=0}^{\infty}\binom{\propto}{k} D_t^{\propto-k}[t^0]D_t^k[h(t)+g(t)] = D_t^\propto[h(t)]+D_t^\propto[g(t)]$.

- $D_t^\propto[h(at)] = a^\propto D_x^\propto[h(x)]$under the scaling $x = at$.

- $D_t^\propto[t^{-m}] = (-1)^\propto \frac{\Gamma(m+\propto)}{\Gamma(m)} t^{-(m+\propto)}$for a given $m \in \mathbb{R}$.

- $D_t^{\mu+\nu}[f(t)] = D_t^\mu[D_t^\nu(f(t))] = D_t^\nu[D_t^\mu(f(t))]$ under the composition of D_t^ν and D_t^μon $f(t)$.

- $D_t^{-1}\left[t^\beta\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+1)} t^{\beta+1} = \frac{t^{\beta+1}}{\beta+1}$, where $\beta \in \mathbb{R}$ corresponding to a negative order derivative.

2.7 Mittag-Leffler Function:

The Exponential function play a important role in the theory of integer order differential equationitsone parameter generalization is denoted by[4]

$$\boldsymbol{E_\alpha(z)}=\sum_{\boldsymbol{k=0}}^{\infty} \frac{\boldsymbol{z^k}}{\Gamma(\alpha k+1)}$$

was introduced by G.M Mittag Leffler [5,6,7] and also studied by A.William[8,9]**.**

3.Laplace Transforms and Inverse Laplace of Fractional Derivatives:

The Laplace transform of a function f(t) is defined as

F(s) = L(f(t))=$\int_0^{\infty} e^{-st}$ f(t) dt

For existence of Laplace transform of f(t), f(t) must be of exponential order. The original f(t) can be obtained from F(s) with the help of [3]Inverse Laplace Transform

f(t) = $L^{-1}[F(s),t] = \int_{c-i\infty}^{c+i\infty} e^{s} F(s)ds$,   c= Re(s)> c_0

Where $c_0$lies in right half plane of the absolute convergence of Laplace integral.

Laplace Transform of convolution is defined as

f(t)*g(t) =$\int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t g(t-\tau)f(\tau)d\tau$

Another useful property which we are needed is Laplace Transform of derivative of an integer order

n of a function f(t):

L{$f^n(t)$;s}=$s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1}\{f^n(0) = s^n F(s) - \sum_{k=0}^{n-1} s^k f^{n-k-1}(0)$

Likewise, we can easily prove Laplace Transform of Fractional derivatives of order p> 0 in terms of Riemann-Liouville Derivativesp[th]

L{ ${}_0D_t^p f(t); s$}=s^pF(s)$- \sum_{k=0}^{n-1} s^k\ [\ {}_0D_t^{p-k-1} f(t)]$ (n-1$\leq p < n$)

Similarly, we can easily establish Laplace Transform Caputo Derivative as

L{ ${}^{C}D^{\alpha} f(x); s\} = s^p$F(s)$- \sum_{k=0}^{n-1} s^{p-k-1}\{f^n(0)\}$ (n-1$< p \leq n$)

4. Existence and Uniqueness Ressultsof solutions for of initial value problem of the Caputo-Type fractional differential equation:

In this chapter we consider the existence and uniqueness of solutions of initial value problem of fractional order differential equation. First we consider the case of linear fractional order differential equations with continuous coefficients and bring the proof of existence and uniqueness theorem for one –term and n-term fractional differential equations.

Then we give the proof of existence and uniqueness theorem for general term fractional differential equations. Finally,we discuss the dependence of solution of general fractional differential equations on initial conditions.

4.1 Linear Fractional Differential EquationWith Initial Conditions.

In this section the existence and uniqueness of solutions of initial value problem for liner   fractional differential equations with sequential derivatives are discussed.

Let's consider the following initial value problem:

${}_0D_t^{\sigma_n} y(t) + \sum_{j=1}^{n-1} p_j(t)\,[\,{}_0D_t^{\sigma_{n-j}} y(t)] + p_n(t)y(t) = f(t)$ (0$< t < T <\propto$) (4.1.1)

$[\,{}_0D_t^{\sigma_n -1} y(t)]_{t=0}$= b_k ,k=1,2,3……………………………………n,   (4.1.2)

Where $\quad {}_{\alpha}D_t^{\sigma_k} = {}_{\alpha}D_t^{\alpha_k}\,{}_{\alpha}D_t^{\alpha_{k-1}} \dots\dots\dots\, {}_{\alpha}D_t^{\alpha_1}$

${}_{\alpha}D_t^{\uparrow\sigma_k-1} = {}_{\alpha}D_t^{\uparrow\alpha_k-1}\,{}_{\alpha}D_t^{\alpha_{k-1}} \dots\dots\dots\dots\dots\dots\, {}_{\alpha}D_t^{\alpha_1}$

Where$\sigma_k = \sum_{j=i}^{k} \alpha_j$ (k=1.2…………………..n)

$0< \alpha_j \leq 1\ (j = 1,2 \dots\dots\dots n)$

And f(t) $\in L_1(0,T)$ i.e.$\int_0^T |f(t)| < \infty$

Here we assume f(t)=0 for t$> T$. Also we can have p_k(t)=0 for k=1,2……….n

4.1.1 Theorem:If$(t) \in L_1(0,T)$, Then the equation ${}_0D_t^{\sigma_n} y(t) = f(t)$ (4.1.1.1)

has unique solution$y(t) \in L_1(0,T)$,which satisfying the initial conditions given by (4.1.2)

Proof: Using Laplace transform of Sequential Fractional Derivative and equation (4.1.1.1), we get

s^{σ_n} Y(s) +$\sum_{k=0}^{n-1} s^{\alpha_n-\sigma_{n-k}} [\,[\,{}_0D_t^{\sigma_{n-k}-1} y(t)]_{t=0}$= F(s)

Where,Y(s) and F(s) denote the Laplace transform of y(t) and f(t).

Using Initial Conditions:

Y(s)=$s^{-\sigma_n}$ F(s) +$\sum_{k=0}^{n-1} b_{n-k} s^{-\sigma_{n-k}}$ and by Inverse Laplace Transform

y(t)=1/$\Gamma(\sigma_n) \int_0^t (t-\tau)^{\sigma_n-1}f(\tau)d\tau + \sum_{k=0}^{n-1} \frac{b_{n-k}}{\Gamma(\sigma_{n-k})} t^{\sigma_{n-k}-1}$

For $n-k=i$ we havey(t)=1/$\Gamma(\sigma_n) \int_0^t (t-\tau)^{\sigma_n-1}f(\tau)d\tau + \sum_{i=1}^{n} \frac{b_i}{\Gamma(\sigma_i)} t^{\sigma_i-1}$

Now by definition of Riemann-Liouville Derivatives of power function and taking in to account that $\frac{1}{\Gamma(-m)} = 0$ for $m = 1,2,3 \dots ....$ we can easily obtained

${}_0D_t^{\sigma_k}(\frac{t^{\sigma_i-1}}{\Gamma(\sigma_i)}) = \frac{t^{\sigma_i-\sigma_k-1}}{\Gamma(\sigma_i-\sigma_k)}$,k$< i$ and ${}_0D_t^{\sigma_k}(\frac{t^{\sigma_i-1}}{\Gamma(\sigma_i)})$=0 when$k \geq i$

${}_0D_t^{\sigma_k}(\frac{t^{\sigma_i-1}}{\Gamma(\sigma_i)}) = \frac{t^{\sigma_i-\sigma_k}}{\Gamma(\sigma_i-\sigma_k-1)}$,k$< i$ and ${}_0D_t^{\sigma_k}(\frac{t^{\sigma_i-1}}{\Gamma(\sigma_i)})$=1when $i = k$

${}_0D_t^{\sigma_k}(\frac{t^{\sigma_i-1}}{\Gamma(\sigma_i)})$=0 if k$> i$ Where k= 1,2,………n and i=1,2,3…………………..n

It follows that y(t) $\in L_1(0,T)$ and it satisfies the initial conditions. Soexistence of solution is proved. Uniqueness follows from the linear property of fractional derivative and Laplace Transform.

Indeed If there exist two solutions y_1(t)and y_2(t) of considered problem,then the function

z(t)= $y_1(t) - y_2(t)$ must satisfies the ${}_0D_t^{\sigma_n} z(t) = 0$ and the initial conditions which gives Laplace transform of z(t) as zero and it proves the uniqueness of the solution.

4.1.2 Theorem:If$f(t) \in L_1(0,T)$ and p_j(t) (j=1,2,3……………… n) are continuous functions in the closed interval [0,T] Then the initial value problem (4.1.1-4.1.2) has a unique solution $y(t) \in L_1(0,T)$.

Proof:Let us assume that the above equation has solution y(t) and,let us consider

${}_0D_t^{\sigma_n} y(t) = \varphi(t)$

Using previous theorem

y(t)=1/Γ(σ_n) $\int_0^t (t-\tau)^{\sigma_n -1} f(\tau)d\tau + \sum_{i=1}^{n} \frac{b_i}{\Gamma(\sigma_i)} t^{\sigma_i -1}$ (4.1.2.1)

using 4.1.1 and 4.1.2.1 we have ${}_0D_t^{\sigma_n} y(t) + \sum_{k=1}^{n-1} p_{n-k}(t)\ {}_0D_t^{\sigma_k} y(t) + p_n(t)y(t) = f(t)$

We obtain the volterra integral equation for the function $\varphi(t)$:

$$\varphi(t) + \int_0^t K(t,\tau)\,\varphi(t)dt = g(t)$$

Where $K(t,\tau) = p_n(t)\frac{(t-\tau)^{\sigma_n -1}}{\Gamma(\sigma_n)} + \sum_{k=1}^{n-1} p_{n-k}(t)\frac{(t-\tau)^{\sigma_n-\sigma_k-1}}{\Gamma(\sigma_n-\sigma_k)}$

$$g(t) = f(t) - p_n(t)\sum_{i=1}^{n} b_i \frac{t^{\sigma_i -1}}{\Gamma(\sigma_i)} - \sum_{k=1}^{n-1} p_{n-k}(t) \sum_{i=k+1}^{n} b_i \frac{t^{\sigma_i-\sigma_k-1}}{\Gamma(\sigma_i-\sigma_k)}$$

As p_j(t) (j=1,2,3……………… n) are continous functions in the closed interval [0,T]

So we have

$K(t,\tau) = \frac{k^*(t,\tau)}{(t-\tau)^{1-\mu}}$ and

Where $k^*(t,\tau)$is continous for $0 \le t \le T$ and $0 \le \tau \le T$ and

$$\mu = \min(\sigma_n, \sigma_n - \sigma_{n-1}, \sigma_n - \sigma_{n-2}, \sigma_n - \sigma_{n-3} \dots\dots\dots\dots\dots\dots\dots\dots .. \sigma_n - \sigma_1) = \min(\sigma_n, \alpha_n)$$

Similarly, g(t) can be written

$$g(t) = \frac{g^*(t)}{(t)^{1-\nu}}$$

Where $g^*(t)$is continous in [0,T] and

$\nu = \min(\sigma_1, \sigma_2, \sigma_3 \dots .. \sigma_n; \sigma_2 - \sigma_1, \dots\dots\dots\dots\dots\dots \sigma_n - \sigma_1; \dots\dots .. \sigma_n - \sigma_{n-1}) = \min(\sigma_n, \alpha_n)$

clearly$0 < \mu \le 1$ and$0 < \nu \le 1$ .It is known that the equation with weak singular kernel and with choice of g(t) has a unique solution $\varphi(t) \in L_1(0,T)$.The unique solution (t) $\in L_1(0,T)$ can be obtained using the previous theorem.

4.1.3Fractional Differential Equationof a General Form

In this section we discuss the existence and uniqueness of a solution of an initial-value problem forthe fractional differential equation of a general form in terms of the Miller- Ross sequential Fractional fractional derivatives . Due to the link between the Miller -Ross, the Rieman-Liouville, the Griinwald-Letnikov, and the Caputo fractional derivatives, the results given below can be usedfor all these mutations of fractional differentiation.

Let us consider the initial-value problem

${}_0D_t^{\sigma_n} y(t) = f(t,y)$ **(4.1.3.1)**

$[\,{}_0D_t^{\sigma_n -1} y(t)]_{t=0} = b_k$,k=1,2,3…………………………………..n, **(4.1.3.2)**

Where $\quad {}_{\alpha}D_t^{\sigma_k} = {}_{\alpha}D_t^{\alpha_k}\, {}_{\alpha}D_t^{\alpha_{k-1}} \dots\dots\dots.. {}_{\alpha}D_t^{\alpha_1}$

${}_{\alpha}D_t^{\uparrow\sigma_k-1} = {}_{\alpha}D_t^{\uparrow\alpha_k-1}\, {}_{\alpha}D_t^{\alpha_{k-1}} \dots\dots\dots\dots\dots\dots {}_{\alpha}D_t^{\alpha_1}$

Where$\sigma_k = \sum_{j=i}^{k} \alpha_j$ (k=1.2…………………..n)

0< $\alpha_j \leq 1$ $(j = 1,2 \dots\dots\dots..n)$

Let us suppose that $f(t,y)$ is defined in a domaim G of plane (t,y) and defined a region $R(h,K) \subset G$ as a set of points $(t,y) \in G$ which satisfy the following inequalities:

0< $t < h$,

$$\left|t^{1-\sigma_1}y(t) - \sum_{i=1}^{n} \frac{t^{\sigma_i-\sigma_1}b_i}{\Gamma(\sigma_i)}\right| \leq K \qquad \textbf{(4.1.3.3)}$$

Where h and K are constant.

Theorem4.1.3.1 Let $f(t,y)$ be a real valued continuous function ,defined in the domain G, satisfying in G the Lipschitz condition with respect to y ,i.e.

$$|f(t,y_1) - f(t,y_2)| \leq A|y_1 - y_2|$$

Such that

$|f(t,y)| \leq M < \infty$ for all$(t,y) \in G$

Let also

$$K \geq \frac{Mh^{\sigma_n-\sigma_1+1}}{\Gamma(1+\sigma_n)}$$

Then there exists a unique and continous solution y(t) of the problem (**4.1.3.1**)-(**4.1.3.2**) in the region $R(h,K)$.

Proof: We first reduce the the problem problem (**4.1.3.1**)-(**4.1.3.2**) to an equivalent fractional integral equation .so performing subsequently the fractional integration of order $\alpha, \alpha_{n-1} \dots\dots\dots\dots\dots\dots\dots\dots\dots. \alpha_1$ we obtain

$$y(t) = \sum_{i=1}^{n} \frac{b_i}{\Gamma(\sigma_i)} t^{\sigma_i-1} + \frac{1}{\Gamma(\sigma_n)} \int_0^t (t-\tau)^{\sigma_n-1} f(\tau, y(\tau))\,d\tau. \qquad \textbf{(4.1.3.3)}$$

we see that if y(t) is the solution of the problem (**4.1.3.1**)-(**4.1.3.2**) ,then it also satisfies (**4.1.3.3**) .

Now we define the sequence of functions$y_0(t), y_1(t), y_2(t), \dots\dots\dots\dots\dots\dots\dots\dots\dots.$

by defining the relationship:

$$y_0(t) = \sum_{i=1}^{n} \frac{b_i}{\Gamma(\sigma_i)} t^{\sigma_i-1} \quad , \qquad \textbf{(4.1.3.4)}$$

$$y_m(t) = \sum_{i=1}^{n} \frac{b_i}{\Gamma(\sigma_i)} t^{\sigma_i-1} + \frac{1}{\Gamma(\sigma_n)} \int_0^t (t-\tau)^{\sigma_n-1} f(\tau, y_{m-1}(\tau))\,d\tau, \qquad \textbf{(4.1.3.5)}$$

m= 1,2,3,………..

We will show that $\log_{m\to\infty} y_m$ (t) exists and gives the required solution y(t) of the given equation.

It can be shown that for $0< t < h$ we have

y_m (t) $\in R(h, K)$ for all m and

$$\left| t^{1-\sigma_1} y(t) - \sum_{i=1}^{n} \frac{t^{\sigma_i - \sigma_1} b_i}{\Gamma(\sigma_i)} \right| \leq K$$

Therefore,

$$\left| t^{1-\sigma_1} y_1(t) - \sum_{i=1}^{n} \frac{t^{\sigma_i - \sigma_1} b_i}{\Gamma(\sigma_i)} \right| \leq K$$

And further it can be proved by induction b that for all m

$$|y_m(t) - y_{m-1}(t)| \leq \frac{A^{m-1 t^{m\sigma\, n}} M}{\Gamma(1 + m\sigma_n)}$$

Now for m=1 we have :

$|y_1(t) - y_0(t)| \leq \frac{t^{m\sigma\, n} M}{\Gamma(1+\sigma_n)}$ $(0< t \leq h)$

Let us suppose that

$|y_{m-1}(t) - y_{m-2}(t)| \leq \frac{A^{m-2 t^{(m-1)\sigma_n}} M}{\Gamma(1+m\sigma_n)}$ $(0< t \leq h)$ **(4.1.3.6)**

Now by using **(4.1.3.5)** , **(4.1.3.6)** and the Riemann-Liouville Derivatives , we have

$|y_m(t) - y_{m-1}(t)| \leq \frac{A}{\Gamma(\sigma_n)} \int_0^t (t-\tau)^{\sigma_n |y_{m-1}(\tau) - y_{m-2}(\tau)|} f(\tau, y(\tau)) d\tau.$

$\leq \frac{A^{m-1 t^{m\sigma\, n}} M}{\Gamma(1+m\sigma_n)}$

This proves the series $y_0(t) = \sum_{i=1}^{n} \frac{b_i}{\Gamma(\sigma_i)} t^{\sigma_i - 1}$convergent uniformly and which allow us to take $m \to \infty$.
Showing that y(t) is the solution of the given fractional differential equation given by (**4.1.3.3)**

5. **Existence and UniquenessResult as a Method of Solution**

In this section We use Laplace Method to solve ordinary Fractional differential Equation in Caputo sense.

Example1: Consider the equation

${}_0D_t^{\frac{1}{2}} f(t) + af(t) = 0 \;\; (t > 0); [\, {}_0D_t^{\frac{-1}{2}} f(t)]_{t=0} = C$

Applying Laplace Transform, we get

F(S)=$\frac{C}{s^{1/2}+a}$, $[\, {}_0D_t^{\frac{-1}{2}} f(t)]_{t=0} = C$

Using Inverse Laplace Transform

$$f(t) = Ct^{1/2}E_{\frac{1}{2},\frac{1}{2}}(-a\sqrt{t})$$

Example2: Let us consider the following Equations

$$_0D_t^Q f(t) + D_t^q f(t) = h(t)$$

Let's assume here $0 < q < Q < 1$. Laplace Transform Of above equations leads to

$((s^Q + s^q)F(s) = C + H(s)$

$[\,_0D_t^{Q-1} f(t) + D_t^{q-1} f(t)]_{t=0} = C$ and then after taking inversion for $\beta = Q\ and\ \alpha = Q - q$

we get

$$f(t) = CG(t) + \int_0^t G(t-\tau)h(\tau)d\tau$$

$[\,_0D_t^{Q-1} f(t) + D_t^{q-1} f(t)]_{t=0} = C$

$$G(t) = t^{Q-1}\boldsymbol{E_{Q-q,Q}}(-\boldsymbol{t^{Q-q}})$$

Example 3: Consider the following non-homogenous fractional differential equation

$$_0D_t^\alpha y(t) + \lambda y(t) = h(t)$$

$[\,_0D_t^{\alpha-k} y(t)]_{t=0} = b_k$,k=1,2,3…………………………………..n, where $n-1 < \alpha < n$

Taking into account the initial conditions ,The Laplace Transform of the above equation is

$$s^\alpha Y(s) - \lambda Y(s) = H(s) + \sum_{k=1}^{n} b_k\, s^{k-1}$$

The inverse Laplace Transform give the solution:

Y(t)$y(t) = \sum_{k=1}^{n} b_k\, t^{\alpha-k}\boldsymbol{E_{\alpha,\alpha-k+1}}(\lambda\boldsymbol{t^\alpha}) + \int_0^t (t-\tau)^{\alpha-1}\boldsymbol{E_{\alpha,\alpha}}(\lambda(t-\tau)^{\boldsymbol{\alpha}})h(\tau)d\tau.$

6. Conclusion

This paper has investigated the existence and uniqueness of solution to the Caputo-type fractional differentialequation with general forms. The first sufficient conditionproving existence and uniqueness of the all type of fractional differential equations.d.. At last, examples are provided to illustrate the applications of the abstract results.

7. References

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