

STUDY OF CONVEX SPACES AND THEIR TENSOR PRODUCTS

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ABSTRACT

In this paper we will make considerable use of the notion of a continuous bilinear map $X \times Y \rightarrow Z$ where X, Y and Z are topological vector spaces. In particular, we will make use of the convex spaces and its tensor products.

In this section we will make considerable use of the notion of a continuous bilinear map $X \times Y \rightarrow Z$ where X, Y and Z are topological vector spaces. In particular, we will make use of the following :

1. Proposition

If X, Y and Z are topological vector spaces then any bilinear map $\phi : X \times Y \rightarrow Z$ is continuous if it is continuous at $(0,0)$.

Proof. Let (x_0, y_0) be a point of $X \times Y$ and let W be a 0-neighbourhood in Z . We choose a 0-neighbourhood W_1 with $W_1 + W_1 + W_1 \subset W$. Since ϕ is continuous at $(0,0)$, there are 0-neighbourhood $U \subset X$ and $V \subset Y$ such that $\phi(U \times V) \subset W_1$. If we choose $s, t \in (0,1)$ so that $x_0 \in sU, y_0 \in tV, sU \subset U$ and $tV \subset V$, then for $x \in x_0 + sU$ and $y \in y_0 + tV$ we have

$$\phi(x, y) - \phi(x_0, y_0) = \phi(x - x_0, y_0) + \phi(x - x_0, y - y_0) + \phi(x_0, y - y_0) \subset W_1 + W_1 + W_1 \subset W.$$

This proves that ϕ is continuous at (x_0, y_0) .

The algebraic tensor product $X \otimes y : x \in X, y \in Y$ subject to the relations

$$(ax_1 + bx_2) \otimes y = a(x_1 \otimes y) + b(x_2 \otimes y), x_1, x_2 \in X, y \in Y, a, b \in \mathbb{C}$$

$$x \otimes (ay_1 + by_2) = a(x \otimes y_1) + b(x \otimes y_2), x \in X, y_1, y_2 \in Y, a, b \in \mathbb{C}$$

It follows that every element $u \in X \otimes Y$ may be written as a finite sum

$$u = \sum_{i=1}^n x_i \otimes y_i.$$

The minimal number n of terms required in a representation of u as above is called the rank of u . If u is expressed as above using a minimal number of terms, that is, so that the number of terms n is equal to the rank of u , then it turns out that the sets $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ must both be linearly independent (problem 6). Of course, this representation of u is far from being unique[1-4].

One easily shows that the tensor product is characterized by the following universal property:

2. Proposition

The map $\theta : X \times Y \rightarrow X \otimes Y$, defined by $\theta(x, y) = x \otimes y$, is a bilinear map with the property that any bilinear map $X \times Y \rightarrow Z$ to a vector space Z is the composition of θ with a unique linear map $\psi : X \otimes Y \rightarrow Z$.

If X and Y are locally convex topological vector spaces, there are at least two interesting and useful ways of giving $X \otimes Y$ a corresponding locally convex topology. The most natural of these is the projective tensor product topology, which we describe below :

If p and q are continuous seminorms on X and Y , respectively, we define the tensor product seminorm $p \otimes q$ on $X \otimes Y$ as follows:

$$(p \otimes q)(u) = \inf \left\{ \sum p(x_i)q(y_i) : u = \sum x_i \otimes y_i \right\}$$

It follows easily that $p \otimes q$ is, indeed, a seminorm. Furthermore, we have:

3. Lemma

For seminorms p and q on X and Y .

- (1) $(p \otimes q)(x \otimes y) = p(x)q(y)$ for all $x \in X, y \in Y$;
- (2) If $U = \{x \in X : p(x) < 1\}$ and $V = \{y \in Y : q(y) < 1\}$, then
 $co(\theta(U \times V)) = \{u \in X \otimes Y : (p \otimes q)(u) < 1\}$

Proof. From the definition, it is clear that

$$(p \otimes q)(x \otimes y) \leq p(x)q(y) \text{ for all } x \in X, y \in Y$$

On the other hand, for a fixed $(x, y) \in X \times Y$, using the Hahn-Banach theorem we may choose linear functional f on X and g on Y such that $f(x) = p(x), g(y) = q(y)$ and $|f(x')| \leq p(x'), |g(y')| \leq q(y')$ for all $x', y' \in X \times Y$. Then if $x \otimes y = \sum x_i \otimes y_i$ is any representation of $x \otimes y$ as a sum of rank one tensors, we have

$$p(x)q(y) = f(x)g(y) = \sum f(x_i)g(y_i) \leq \sum p(x_i)q(y_i)$$

Since $(p \otimes q)(x \otimes y)$ is the inf of the expressions on the right side of this inequality we have $p(x)q(y) \leq (p \otimes q)(x \otimes y)$. This proves (1).

Certainly $co(\theta(U \times V)) \subset \{u \in X \otimes Y : (p \otimes q)(u) < 1\}$ since the latter is a convex set containing $\theta(U \times V)$. To prove the reverse containment, let u be an element of $X \otimes Y$ with $(p \otimes q)(u) < 1$. Then we can represent u as $u = \sum x_i \otimes y_i$ with

$$\sum p(x_i)q(y_i) = r^2 < 1$$

If we set $x'_i = r p(x_i)^{-1} x_i$ and $y'_i = r q(y_i)^{-1} y_i$, then $p(x'_i) = r = q(y'_i)$. Thus, $x'_i \in U$ and $y'_i \in V$. Furthermore, if $t_i = r^{-2} p(x_i)q(y_i)$, then

$$u = \sum t_i (x'_i \otimes y'_i) \text{ and } \sum t_i = 1$$

Thus, $u \in co(\theta(U \times V))$ and the proof of (2) is complete [5-7].

4. Definition

The topology on $X \otimes Y$ determined by the family of seminorms $p \otimes q$, as above, will be called the projective tensor product topology. We will denote $X \otimes Y$, endowed with this topology, by $X \otimes Y$.

If $f \in X^*$ and $g \in Y^*$ then we may define a linear functional $f \otimes g$ on $X \otimes Y$ by

$$(f \otimes g) \left(\sum x_i \otimes y_i \right) = \sum f(x_i)g(y_i)$$

One easily checks that this is well defined and linear.

5. Proposition

The projective tensor product topology is a Hausdorff locally convex topology on $X \otimes Y$ with the following properties :

- (1) the bilinear map $\theta : X \times Y \rightarrow X \otimes_{\pi} Y$ is continuous;
- (2) $f \otimes g \in (X \otimes_{\pi} Y)^*$ for each $f \in X^*$ and $g \in Y^*$;
- (3) A neighbourhood base for the topology at 0 in $X \otimes_{\pi} Y$ consists of sets of the form $co(\theta(U \times V))$ where U is a 0- neighbourhood in X and V is a 0 neighbourhood in Y .
- (4) any continuous bilinear map $X \times Y \rightarrow Z$ to a locally convex space Z factors as the composition of θ with a unique continuous linear map $X \otimes_{\pi} Y \rightarrow Z$;

Proof. Lemma 3(1) implies that each $(p \otimes q) \circ \theta$ is continuous at $(0,0)$ and this implies that θ is continuous at $(0,0)$ and, hence, is continuous everywhere.

The continuity of $f \otimes g$ for $f \in X^*$ follows from the fact that $|f|$ and $|g|$ are continuous seminorms on X and Y and $|(f \otimes g)(\sum x_i \otimes y_i)| \leq \sum |f(x_i)| |g(y_i)| \leq \sum |f(x_i)| |g(y_i)| \leq |f| \otimes |g| (\sum x_i \otimes y_i)$. This proves (2).

The fact that the projective topology is Hausdorff follows from (2). In fact, if $u \in X \otimes Y$ then we may write $u = \sum x_i \otimes y_i$, where the set $\{x_i\}$ is linearly independent. Then we may choose $f \in X^*$ such that $f(x_i) \neq 0$ if and only if $i = 1$ and we may choose $g \in Y^*$ such that $g(x_i) \neq 0$. Then the element $f \otimes g \in (X \otimes Y)^*$ has the non-zero value $f(x_1)g(x_1)$ at u . Thus, $U = \{v \in X \otimes Y : |(f \otimes g)(v)| < f(x_1)g(x_1)\}$ is an open set containing 0 but not containing u .

Part (3) is an immediate consequence of Lemma 3.(2)

If $\phi : X \times Y \rightarrow Z$ is a continuous bilinear map, then $\phi = \psi \circ \theta$ for a unique linear map $\psi : X \otimes_{\pi} Y \rightarrow Z$ by Proposition 2. To prove (4) we must show that ψ is continuous. Let W be a convex 0-neighbourhood in Z . Since ϕ is continuous, there exist 0-neighbourhoods U and V in X and Y , respectively, such that $\phi(U \times V) \subset W$. Then the convex hull of $\theta(U \times V)$ is a 0-neighbourhood in $X \otimes Y$ by (3) and it clearly maps into W under ψ . Thus ψ is continuous.

Note that (4) of the proposition says that projective tensor product topology is the strongest locally convex topology on $X \otimes Y$ for which the bilinear map $\theta : X \times Y \rightarrow X \otimes Y$ is continuous.

Note that (4) of the above proposition says that projective tensor product topology is the strongest locally convex topology on $X \otimes Y$ for which determines its topology. Also, if X and Y are metrizable then so is $X \otimes Y$.

If X, Y and Z are locally convex spaces and $\alpha : X \rightarrow Y$ is a continuous linear map then the composition

$$X \times Z \rightarrow Y \times Z \rightarrow Y \otimes_{\pi} Z$$

is a continuous bilinear map $X \times Z \rightarrow Y \otimes_{\pi} Z$ and, by Proposition 5.(4), it factors through a unique continuous linear map $\alpha \otimes id : X \otimes_{\pi} Z \rightarrow Y \otimes_{\pi} Z$. This shows that, for a fixed l.c.s. Z , $(.) \otimes_{\pi} Z$ is a functor from the category of locally convex spaces to itself. Similarly, the projective tensor product is also a functor in its second argument for each fixed l.c.s. appearing in its first argument.

6. Proposition

If $\alpha : X \rightarrow Y$ is a continuous linear open map, then so is $\alpha \otimes id : X \otimes Z \rightarrow Y \otimes Z$

Proof. To show that $\alpha \otimes id$ is open we must that each 0- neighbourhood in $X \otimes Z$ maps to a 0-neighbourhood in $Y \otimes Z$. However, this follows immediately from Proposition 5(3) and the hypothesis that α is an open map.

The space $X \otimes_{\pi} Y$ is generally not complete. It is usually useful to complete it.

7. Definition

The completion of $X \otimes_{\pi} Y$ will be denoted $X \widehat{\otimes}_{\pi} Y$ and will be called the completed projective tensor product of X and Y .

Note that if $\alpha : X \rightarrow Y$ is a continuous linear map, the map $\alpha \otimes id : X \widehat{\otimes}_{\pi} Y \rightarrow Y \widehat{\otimes}_{\pi} Z$. Even under the hypothesis of Proposition 3.6 this map is not generally a surjection. However, we do have:

8. Proposition

If X , Y and Z are Frechet spaces and $\alpha : X \rightarrow Y$ is a surjective continuous linear map, then $\alpha \otimes id : X \widehat{\otimes}_{\pi} Y \rightarrow Y \widehat{\otimes}_{\pi} Z$.

Proof. By the open mapping theorem, the map α is open. Then $\alpha \otimes id$ is open by Proposition 3.6. Since, the topologies of X , Y and Z countable bases at 0 the same in true of $X \otimes_{\pi} Z$ and $Y \otimes_{\pi} Z$. However, an open map between metrizable t.v.s.'s has the property that every Cauchy sequence in the range has a subsequence which is the image of a Cauchy sequence in the domain (Exercise 3.1). Since every point in the completion $X \widehat{\otimes}_{\pi} Z$ is the limit of a Cauchy sequence in $Y \widehat{\otimes}_{\pi} Z$, the result follows.

Obviously, the analogues of Proposition 3.6 and 3.8 with the roles of the left and right arguments reversed are also true.

There are other hypothesis under which the conclusion of the above Proposition is true and we will return to this question when we have developed the tools to prove such results[7-9].

In the case where X and Y are Frechet spaces, elements of the completed projective tensor product $X \widehat{\otimes}_{\pi} Y$ may be represented in a particularly useful form:

9. Proposition

If X and Y are Frechet spaces then each element $u \in X \widehat{\otimes}_{\pi} Y$ may be represented as the sum of a convergent series.

$$u = \sum_{i=1}^{\infty} \lambda_i x_i \otimes y_i$$

where $\sum_{i=1}^{\infty} |\lambda_i| < \infty$ and $\{x_i\}$ and $\{y_i\}$ are sequences converging to 0 in X and Y , respectively.

Proof. Let $\{p_n\}$ and $\{q_n\}$ denote increasing sequences of seminorms generating the topologies of X and Y . If $\{u_n\}$ is a sequence in $X \otimes Y$ converging to u in the projective topology, then we may, by replacing $\{u_n\}$ by an appropriately chosen subsequence, assume that the sequence $\{u_n\}$, where $v_n = u_1$ and $v_n = u_n - u_{n-1}$ for $n > 1$, satisfies

$$r_n(v_n) < n^{-2} 2^{-n} \text{ and } \sum v_n = u$$

It follows that we may write each v_n as a finite sum

$$v_n = \sum_i x'_{ni} \otimes y'_{ni} \text{ with } \sum_i p_n(x'_{ni}) q_n(y'_{ni}) < n^{-2} 2^{-n}$$

If we set

$$x_{ni} = n^{-1}p_n(x_{ni}^{-1}) \quad y_{ni} = n^{-1}p_n(y'_{ni})^{-1}y'_{ni}, \quad \lambda_{ni} = n^2p_n(x_{ni})q_n(y_{ni})$$

then

$$p_n(x_{ni}) = q_n(y_{ni}) = n^{-1}, \quad \sum_i |\lambda_{ni}| < 2^{-n} \text{ and } v_n = \sum_i \lambda_{ni} x_i \otimes y_i$$

and so

$$u = \sum_{n,i} \lambda_{ni} x_i \otimes y_i, \quad \sum_i |\lambda_{ni}| < \infty \text{ and } \lim_n p_m(x_{ni}) = \lim_n p_m(x_{ni}) = 0 \quad \forall m$$

The proof is complete if we reindex $\{x_{ni}\}, \{y_{ni}\}$ and $\{\lambda_{ni}\}$ to form singly indexed sequences.

10. Definition

Let $\mathcal{B}(X, Y)$ denote the space of continuous bilinear forms (continuous bilinear maps to $\mathbb{C}$) on $X \times Y$, Let $B(X, Y)$ denote the space of bilinear forms on $X \times Y$ which are continuous in each variable separately.

11. Proposition

The dual space of $X \widehat{\otimes}_\pi Y$ is naturally isomorphic to $B(X, Y)$.

Proof : If $\theta : X \times Y \rightarrow X \widehat{\otimes}_\pi Y$ is the bilinear map defined by $\theta(x, y) = x \otimes y$ then $f \rightarrow f \circ \theta$ is clearly a linear map of $X \widehat{\otimes}_\pi Y$ to $B(X, Y)$. It is an isomorphism by Prop. 3.5 (4).

We now proceed to the second important way of topologizing the tensor product of two locally convex spaces.

Let X_σ^* denote X^* with its weak-* topology - that is, with the weak topology that X induces on X^* . With a similar meaning for Y^* , we note that $X \otimes Y$ is naturally embedded in $B(X_\sigma^*, Y_\sigma^*)$. In fact, if $u = \sum x_i \otimes y_i \in X \otimes Y$ the map

$$(f, g) \rightarrow \phi_u(f, g) = \sum f(x_i)g(y_i): X^* \times Y^* \rightarrow \mathbb{C}$$

is a separately continuous bilinear form on $X_\sigma^* \times Y_\sigma^*$. If we choose the set $\{x_i\}$ to be linearly independent, then it is easy to see that $\phi_u \neq 0$ if $u \neq 0$. Thus, $u \rightarrow \phi_u$ is an embedding of $X \otimes Y$ into $B(X_\sigma^*, Y_\sigma^*)$.

12. Definition

The topology of bi-equicontinuous convergence on $B(X_\sigma^*, Y_\sigma^*)$ is the topology of uniform convergence on sets of the form $A \times B$ where A and B are equicontinuous subsets of X^* and Y^* , respectively [9-11]. We denote by $B_e(X_\sigma^*, Y_\sigma^*)$ the space $B(X_\sigma^*, Y_\sigma^*)$ with this topology. We denote by $X \otimes_e Y$ the space $X \otimes Y$ with the topology it inherits from its natural embedding in $B_e(X_\sigma^*, Y_\sigma^*)$. Finally, we denote by $X \widehat{\otimes}_e Y$ the completion of $X \otimes_e Y$.

Note that a family of seminorms determining the topology on $B_e(X_\sigma^*, Y_\sigma^*)$ consists of the seminorms of the form $p_{A,B}$, where A and B are equicontinuous subsets of X^* and Y^* , respectively and :

$$p_{A,B}(\phi) = \sup\{|\phi(f, g)| : (f, g) \in A \times B\}$$

A typical 0- neighbourhood in this topology has the form

$$V_{A,B} = \{\phi \in B(X_\sigma^*, Y_\sigma^*) : |\phi(f, g)| < 1 \forall f \in A, g \in B\}$$

For this to make sense, we need to know that each $\phi \in B(X_\sigma^*, Y_\sigma^*)$ is bounded on each set of the form $A \times B$ with A and B equicontinuous. However, note that every equicontinuous subset of X^* is contained

in one of the form V^0 for V a-neighbourhood and sets of the form V^0 are compact in X_σ^* by the Banach-Alaoglu theorem, thus, we may assume that A and B are compact and convex. Since an element $\phi \in B(X_\sigma^*, Y_\sigma^*)$ is separately continuous, it is bounded on $\{x\} \times B$ for each $x \in A$ and on $a \times \{y\}$ for each $y \in B$. It follows from the Banach-Steinhaus Theorem for compact convex sets that ϕ is bounded on $A \times B$. Thus, the seminorm $p_{A,B}$ is defined on all of $B(X_\sigma^*, Y_\sigma^*)$. Clearly, the topology generated by the family of such seminorms makes $B_e(X_\sigma^*, Y_\sigma^*)$ into an l.c.s.

If $\alpha : X \rightarrow Y$ is a continuous linear map between l.c.s.'s and Z is any l.c.s. then the adjoint map $\alpha^* : Y_\sigma^* \rightarrow X_\sigma^*$ is also continuous and maps equicontinuous sets to equicontinuous sets. It follows that $\alpha^* : Y_\sigma^* \rightarrow X_\sigma^*$ is also continuous and maps equicontinuous sets to equicontinuous sets. It follows that α^* induces a continuous linear map

$$(\alpha^* \times id)' : B_e(X_\sigma^*, Z_\sigma^*) \rightarrow B_e(Y_\sigma^*, Z_\sigma^*)$$

where

$$(\alpha^* \times id)' \phi(f, g) = \phi(\alpha^*(f), g)$$

Restricted to the image of $X \otimes Z$ in $B_e(X_\sigma^*, Z_\sigma^*)$, this map is just $\alpha \otimes id : X \otimes Z \rightarrow Y \otimes Z$. Thus, it follows that $\alpha \otimes id$ is continuous linear map from $X \otimes_e Z$ to $Y \otimes_e Z$ and extends to a continuous linear map $\alpha \otimes id : X \widehat{\otimes}_e Z \rightarrow Y \widehat{\otimes}_e Z$ between their completion. Thus, for a fixed l.c.s. Z , $(.) \otimes_e Z$ are functions from the category of locally convex spaces to itself.

13. Lemma

If X and Y are l.c.s.'s and $\alpha : X \rightarrow Y$ is a topological isomorphism onto its range. Then under $\alpha^* : Y^* \rightarrow X^*$ each equicontinuous set in X^* is the image of an equicontinuous set in Y^* .

Proof. Since α is a topological isomorphism onto its image, we may identify X with its image and consider it a subspace of Y . Then α is the inclusion and α^* the restriction map from functional on Y to functional on X . Thus, we must show that each equicontinuous set in X^* is the restriction of an equicontinuous set in Y^* . Since the class of equicontinuous sets is closed under passing to subsets, it is enough to show that each equicontinuous set A in X^* is contained in the restriction of an equicontinuous set B in Y^* . Furthermore, we may assume that $A = U^0$ for a convex, balanced 0-neighbourhood $U \subset X$ since every equicontinuous set is contained in one of this form. However, by Lemma 2.10, for each such neighbourhood U there is a convex, balanced 0 neighbourhood $V \subset Y$ such that $V \cap X = U$. The proof will be complete if we can establish that U^0 is the image of V^0 under the restriction map. To this end, we let p_V be the Minkowski functional of V . Since $V \cap X = U$, the restriction of p_V to X is the Minkowski functional p_U of U . Now if $f \in X^*$, it is easy to see that $f \in U^0$ if and only if $|f| \leq p_U$ (Problem 3.5). Thus, if $f \in U^0$ if and only if $|f| \leq p_U$ (Problem 3.5.) Thus, if $f \in U^0$ then $|f| \leq p_V$. Now the Hahn-Banach theorem implies that f has an extension g to Y which satisfies $|g| \leq p_V$. This, in turn, implies that $g \in V^0$. Thus, U^0 is the restriction to X of V^0 and the proof is complete[11-13].

14. Proposition

If X, Y and Z are l.c.s.'s and $\alpha : X \rightarrow Y$ is a topological isomorphism onto its image, then the induced maps $\alpha \otimes_e id : X \otimes Z \rightarrow Y \otimes_e Z$ and $\alpha \otimes id : X \widehat{\otimes}_e Z \rightarrow Y \widehat{\otimes}_e Z$ are as well.

Proof. This will certainly follow if we can show that the above map.

$$(\alpha^* \times id)' : B_e(X_\sigma^*, Z_\sigma^*) \rightarrow B_e(Y_\sigma^*, Z_\sigma^*)$$

is a topological isomorphism onto its range. However, the fact that α is a topological isomorphism onto its range implies that every equicontinuous set in X is the image of an equicontinuous set in Y under α^* by the previous lemma. This, in turn, implies that the 0-neighbourhood in $B_e(Y_\sigma^*, Z_\sigma^*)$ consisting of elements bounded by one on $A \times C$ for a pair of equicontinuous sets $A \subset X^*$ and $C \subset Z$, will be the inverse image under $(\alpha^* \times id)'$ of the 0-neighbourhood, defined analogously, by $B \times C$, where B is an equicontinuous set in Y^* with $\alpha^*(B) = A$. This gives the result for the uncompleted tensor product. The result for the completed tensor products follows from Problem 4.

How are the projective and bi-equicontinuous convergence tensor product topologies related? The projective is stronger. That is,

15. Proposition

The identity map $X \otimes_e Y$ has the form

$$W = \{u \in X \otimes Y : |(f \otimes g)(u)| \leq 1 \forall f \in A, g \in B\}$$

where A and B are equicontinuous subsets of X^* and Y^* , respectively. But if $U = A^0$ and $V = B^0$, then U and V are 0-neighbourhoods in X and Y , respectively, and $\theta(U \times V) \subset W$. Since $co(\theta(U \times V))$ is a 0-neighbourhood in $X \times_\pi Y$, the proof is complete [14].

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